

Sharper estimates for Chebyshev's functions ϑ and ψ

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Abstract

In this article we present some improved results for Chebyshev's functions ϑ and ψ using the new zero-free region obtained by H. Kadiri and the calculated the first 10^{13} zeros of the Riemann zeta function on the critical line by Xavier Gourdon. The methods in the proofs are similar to those of Rosser-Shoenfeld papers on this subject.

1 Chebyshev's functions

Definition 1.1. For $x > 0$ we define Chebyshev's ψ -function by the formula

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

where

$$\Lambda(n) = \begin{cases} \log p & , \quad n = p^m \text{ for some } m; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Since $\Lambda(n) = 0$ unless n is a prime power, we can write the definition of $\psi(x)$ as follows:

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p^m \leq x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p \leq x^{1/m}} \log p. \quad (1.1)$$

The sum on m is actually a finite sum. In fact, the sum on p is empty if $x^{1/m} < 2$, that is, if $(1/m) \log x < \log 2$, or if

$$m > \frac{\log x}{\log 2} = \log_2 x.$$

Therefore, we have

$$\psi(x) = \sum_{m \leq \log_2 x} \sum_{p \leq x^{1/m}} \log p = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p.$$

This can be written in a slightly different form by introducing another function of Chebyshev.

Definition 1.2. If $x > 0$, we define Chebyshev's ϑ -function by the equation

$$\vartheta(x) = \sum_{p \leq x} \log p.$$

The last formula for $\psi(x)$ can now be restated as follows:

$$\psi(x) = \sum_{m \leq \log_2 x} \vartheta(x^{1/m}). \quad (1.2)$$

Using Möbius inversion formula

$$\begin{aligned} \sum_{k \geq 1} \mu(k) \psi(x^{1/k}) &= \sum_{k \geq 1} \mu(k) \sum_{l \geq 1} \vartheta(x^{1/kl}) = \sum_{n \geq 1} \sum_{k|n} \mu(k) \vartheta(x^{1/n}) \\ &= \sum_{n \geq 1} \delta_{1,n} \vartheta(x^{1/n}) = \vartheta(x), \end{aligned}$$

where

$$\mu(n) = \begin{cases} 1 & , \quad n = 1; \\ (-1)^k & , \quad n = \text{product of } k \text{ distinct primes}; \\ 0 & , \quad \text{otherwise,} \end{cases}$$

and $\delta_{i,j}$ is Kronecker's delta function

$$\delta_{i,j} = \begin{cases} 1 & , \quad i = j; \\ 0 & , \quad i \neq j. \end{cases}$$

Theorem 1.3 ([1], p. 76). *For $x > 0$ we have*

$$0 \leq \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \leq \frac{1}{2 \log 2} \frac{\log^2 x}{\sqrt{x}}.$$

Note that this inequality implies that

$$\lim_{x \rightarrow \infty} \left(\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \right) = 0.$$

In other words, if one of $\psi(x)/x$ or $\vartheta(x)/x$ tends to a limit then so does the other, and the two limits are equal.

1.1 Relations connecting $\vartheta(x)$ and $\pi(x)$

In 1896 J. Hadamard and C. J. de la Vallée Poussin independently and almost simultaneously succeeded in proving that

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1.$$

This remarkable result is called the prime number theorem, and its proof was one of the crowning achievements of analytic number theory.

In this section we give two formulas relating $\vartheta(x)$ and $\pi(x)$. These can be used to show that the prime number theorem is equivalent to the limit relation

$$\lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} = 1.$$

Theorem 1.4 (Abel's identity). *For any arithmetical function $a(n)$ let*

$$A(x) = \sum_{n \leq x} a(n),$$

where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then we have

$$\sum_{y < n \leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt. \quad (1.3)$$

Now we use (1.3) to express $\vartheta(x)$ and $\pi(x)$ in terms of integrals.

Theorem 1.5 ([1], p. 78). *For $x \geq 2$ we have*

$$\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt, \quad (1.4)$$

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt. \quad (1.5)$$

1.2 Relations connecting $\psi(x)$ and $\Pi(x)$

From Euler's identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2}^{\infty} \frac{1}{1-1/p^s}, \quad (\Re s > 1).$$

Taking logarithm

$$\log \zeta(s) = - \sum_p \log \left(1 - \frac{1}{p^s} \right) = \sum_{p,m} \frac{1}{mp^{ms}}.$$

By differentiation,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}},$$

or

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

On the other hand,

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

So, by Abel's identity

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} dx, \quad (\Re s > 1).$$

The function

$$\Pi(x) = \sum_{p^m \leq x} \frac{1}{m} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots$$

So

$$\log \zeta(s) = \sum_{p,m} \frac{1/m}{p^{ms}} = \int_1^{\infty} \frac{d\Pi(x)}{x^s} dx = s \int_1^{\infty} \frac{\Pi(x)}{x^{s+1}} dx, \quad (\Re s > 1).$$

Now the connection between Π and ψ .

$$\begin{aligned} \Pi(x) &= \sum_{p^m \leq x} \frac{1}{m} = \sum_{2 \leq n \leq x} \frac{\Lambda(n)}{\log n} \\ &= \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \log^2 t} dt, \end{aligned}$$

Recall that

$$\text{li}(x) = \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right\}.$$

Also

$$\begin{aligned} \text{li}(x) &= \text{li}(2) + \int_2^x \frac{dt}{\log t} = \text{li}(2) + \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{t}{t \log^2 t} dt, \\ \text{li}(2) &\approx 1.04516, \quad \frac{2}{\log 2} \approx 2.88539. \end{aligned}$$

So

$$\Pi(x) - \text{li}(x) = \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(t) - t}{t \log^2 t} dt + O(1). \quad (1.6)$$

Theorem 1.6 ([1], p. 79). *The following relations are logically equivalent:*

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1, \quad (1.7)$$

$$\lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} = 1, \quad (1.8)$$

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1. \quad (1.9)$$

1.3 Chebyshev's functions and the Riemann zeta function

The zeta function was introduced in mathematics as an analytic tool for studying prime numbers. Therefore, it is only natural that some of the most important applications of the zeta function belong to prime number theory. Here we shall be concerned with some of the most important of these applications.

Many problems in prime number theory may be formulated in terms of the functions π , ϑ , and ψ .

Proposition 1.7. *We have*

$$\frac{\psi(x)}{\log x} < \pi(x) < \int_2^x \frac{d\psi(t)}{\log t}. \quad (1.10)$$

Proof. Since

$$\begin{aligned} \pi(x) &= \sum_{p \leq x} 1 = \sum_{p \leq x} \frac{1}{[\log x / \log p] \log p} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \\ &= \int_2^x \frac{1}{[\log x / \log t] \log t} d\psi(t). \end{aligned} \quad (1.11)$$

and

$$1 \leq \left\lfloor \frac{\log x}{\log t} \right\rfloor \leq \frac{\log x}{\log t}$$

□

Already Riemann, whose work was in many aspects decades beyond that of his contemporaries, stated the elegant formula, which says that the weighted function ψ is in a certain sense more natural than π and ϑ , since it possesses a (relatively simple) explicit expression, and relates the order of $\psi(x) - x$ to a certain sum over non-trivial zeros of the zeta function; namely

$$\boxed{\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - \frac{1}{x^2}), \quad (x > 1, x \neq p^m),} \quad (1.12)$$

where $\rho = \beta + i\gamma$ is a non-trivial zero of $\zeta(s)$, and

$$\sum_{\rho} \frac{x^{\rho}}{\rho} = \lim_{T \rightarrow \infty} \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho}.$$

and when $x = p^m$, then in the left-hand side of (1.12) put $\psi(x) - \frac{1}{2}\Lambda(x)$. This explicit expression for $\psi(x)$ was proved by H. von Mangoldt in 1895.

1.4 The error term in the prime number theorem

The size of the error term in the prime number theorem depends on the location of zeros of the Riemann zeta function [10]. If

$$\{s = \sigma + it : \sigma > 1 - \frac{c}{\log |t|}, |t| > t_0\}$$

is a zero free region, then an explicit error term in the prime number theorem is

Theorem 1.8 ([5], p. 141). *There exists a constant $a > 0$ such that for x tending to infinity, we have*

$$\begin{aligned}\psi(x) - x &= O(x \exp(-a\sqrt{\log x})) \\ \vartheta(x) - x &= O(x \exp(-a\sqrt{\log x})) \\ \Pi(x) - \text{Li}(x) &= O(x \exp(-a\sqrt{\log x})) \\ \pi(x) - \text{Li}(x) &= O(x \exp(-a\sqrt{\log x}))\end{aligned}$$

One can choose $a = 1/15$ and all constants “O” are effective.

Theorem 1.9 ([5], p. 425). *There exist a positive constant α such that for x infinity we have*

$$\pi(x) - \text{Li}(x) = O \left\{ x \exp \left(-\alpha \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \right\}$$

The constant “O” is effective and can take $\alpha = 0.009$. The corresponding asymptotic formulas take place for $\vartheta(x)$, $\psi(x)$ and $\Pi(x)$.

Cheng [2] gives an explicit zero-free region for the Riemann zeta-function derived from the Vinogradov- Korobov method. He proves that the Riemann zeta-function does not vanish in the region

$$\sigma \geq 1 - \frac{0.00105}{\log^{2/3} |t| (\log \log |t|)^{1/3}}, \quad |t| \geq 3$$

In turn, he shows using these results that for all $x > 10$

$$|\pi(x) - \text{li}(x)| \leq 11.88x(\log x)^{3/5} \exp\left(-\frac{1}{57}(\log x)^{3/5}(\log \log x)^{1/5}\right)$$

and for $x \geq e^{44.08}$, there is a prime between x^3 and $(x+1)^3$.

Theorem 1.10 ([6]). *If*

$$|\zeta(\sigma + it)| \leq A|t|^{B(1-\sigma)^{3/2}} \log^{2/3} |t|, \quad \left(\frac{1}{2} \leq \sigma \leq 1, |t| \geq 3, A = 76.2\right)$$

holds with a certain constant B , then for large $|t|$, $\zeta(\sigma + it) \neq 0$ for

$$\sigma \geq 1 - \frac{0.05507B^{-2/3}}{(\log |t|)^{2/3}(\log \log |t|)^{1/3}}$$

Taking $B = 4.45$ gives the zero-free region.

1.5 The results of Ingham

Theorem 1.11 ([8], p. 66). *When $x \rightarrow \infty$,*

$$\psi(x) = x + O(x \exp(-a\sqrt{\log x \log \log x})) \quad (1.13)$$

$$\pi(x) = \text{li}(x) + O(x \exp(-a\sqrt{\log x \log \log x})) \quad (1.14)$$

where a is a positive absolute constant.

Let Θ be the upper bound of the real parts of the zeros of $\zeta(s)$. Clearly $\Theta \leq 1$, since there is no zeros in $\sigma > 1$. And from the existence of the non-trivial zeros ρ and their symmetry about the line $\sigma = \frac{1}{2}$ we infer that $\Theta \geq \frac{1}{2}$. Thus $\frac{1}{2} \leq \Theta \leq 1$, and this is the most that is known about Θ ; but $\Theta = \frac{1}{2}$ if (and only if) the Riemann hypothesis is true. We now have the following theorem, which is worthless if $\Theta = 1$.

Theorem 1.12 ([8], p. 83).

$$\psi(x) = x + O(x^\Theta \log^2 x)$$

$$\pi(x) = \text{li}(x) + O(x^\Theta \log x)$$

Theorem 1.13 ([8], p. 90). *If δ is any fixed positive number, then*

$$\psi(x) - x = \Omega_\pm(x^{\Theta-\delta})$$

$$\Pi(x) - \text{li}(x) = \Omega_\pm(x^{\Theta-\delta})$$

Theorem 1.14 ([8], p. 100). *We have*

$$\psi(x) - x = \Omega_\pm(x^{1/2} \log \log \log x)$$

when $x \rightarrow \infty$. In fact,

$$\limsup \frac{\psi(x) - x}{x^{1/2} \log \log \log x} \geq \frac{1}{2}$$

$$\liminf \frac{\psi(x) - x}{x^{1/2} \log \log \log x} \leq -\frac{1}{2}$$

2 New Explicit Bounds for Some Functions of Prime Numbers

Riemann Hypothesis verified until the 10^{13} -th zero by Gourdon (October 12th 2004) [7].

Recall $N(T)$, $F(T)$ and $R(T)$ be defined as

$$N(T) = \#\{\rho : \zeta(\rho) = 0, 0 < \gamma \leq T\} \quad (2.1)$$

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} \quad (2.2)$$

$$R(T) = 0.137 \log T + 0.443 \log \log T + 1.588 \quad (2.3)$$

Theorem 2.1 ([12]). *For $T \geq 2$,*

$$|N(T) - F(T)| < R(T)$$

Choose A such that $F(A) = 10^{13}$. Then

$$A = 2,445,999,556,030.342,362,641$$

$$\log A = 28.525,474,972.$$

Lemma 2.2 ([11]). *We have*

$$\sum_{\rho} \frac{1}{\gamma^2} < 0.0463.$$

Proposition 2.3.

$$\sum_{\rho} \frac{1}{|\gamma^3|} < 0.00146435,$$

$$\sum_{\rho} \frac{1}{\gamma^4} < 7.43617 \cdot 10^{-5},$$

$$\sum_{\rho} \frac{1}{|\gamma^5|} < 4.46243 \cdot 10^{-6},$$

$$\sum_{\rho} \frac{1}{\gamma^6} < 2.88348 \cdot 10^{-7},$$

$$\sum_{\rho} \frac{1}{|\gamma^7|} < 1.93507 \cdot 10^{-8}.$$

Proof. We use the same method as in [11], and letting $r = 29$ instead of 8. □

Theorem 2.4 ([9]). *The Riemann zeta-function $\zeta(s)$ doesn't vanish in the region*

$$\sigma \geq 1 - \frac{1}{R_0 \log |t|}, \quad (|t| \geq 2, R_0 = 5.69693)$$

In other words, if $\rho = \beta + i\gamma$ is a zero of Riemann zeta function, then

$$\beta < 1 - \frac{1}{R_0 \log |t|}, \quad (|t| \geq 2, R_0 = 5.69693)$$

2.1 Estimates for certain integrals to the Bessel functions

Let

$$K_{\nu}(z, x) = \frac{1}{2} \int_x^{\infty} t^{\nu-1} H^z(t) dt$$

where $z > 0$, $x \geq 0$ and

$$H^z(t) = \exp\left\{-\frac{1}{2}z\left(t + \frac{1}{t}\right)\right\}$$

Lemma 2.5 ([14]).

$$K_{\nu}(z, x) + K_{-\nu}(z, \frac{1}{x}) = K_{\nu}(z, 0) = K_{\nu}(z)$$

$$K_1(z, x) < \frac{e^{-z}}{2z} \left\{ \left(1 + \frac{3\sqrt{2}y}{8}\right) e^{-zy^2} + \left(\frac{3}{8} + z\right) \sqrt{2} \int_y^{\infty} e^{-zw^2} dw \right\} \quad (2.4)$$

$$K_2(z, x) < \frac{e^{-z}}{2z} \left\{ \left[\frac{35\sqrt{2}}{64} y^3 + 2y^2 + \left(\frac{105}{128z} + \frac{15}{8}\right) \sqrt{2}y + 2 + \frac{2}{z} \right] e^{-zy^2} + \left(\frac{105}{128z} + \frac{15}{8}\right) \sqrt{2} \int_y^{\infty} e^{-zw^2} dw \right\} \quad (2.5)$$

where $y = (\sqrt{x} - 1/\sqrt{x})/\sqrt{2}$. If we let x go to 0, then

$$K_1(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{3}{8z}\right) \quad (2.6)$$

$$K_2(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{15}{8z} + \frac{105}{128z^2}\right) \quad (2.7)$$

2.2 Bounds for $\psi(x) - x$ for Large Values of x

Lemma 2.6 ([14]). *Let $1 < U \leq V$, and let $\Phi(y)$ be non-negative and differentiable for $U < y < V$. Let $(W - y)\Phi'(y) \geq 0$ for $U < y < V$, where W need not lie in $[U, V]$. Let Y be one of U, V, W which is neither greater than both the others nor less than both the others. Choose $j = 0$ or 1 so that $(-1)^j(V - W) \geq 0$. Then*

$$\begin{aligned} \sum_{U < \gamma \leq V} \Phi(\gamma) &\leq \frac{1}{2\pi} \int_U^V \Phi(y) \log \frac{y}{2\pi} dy \\ &\quad + (-1)^j \left\{ 0.137 + \frac{0.443}{\log Y} \right\} \int_U^V \frac{\Phi(y)}{y} dy + E_j(U, V) \end{aligned}$$

where the error term $E_j(U, V)$ is given by

$$\begin{aligned} E_j(U, V) &= \{1 + (-1)^j\} R(Y) \Phi(Y) \\ &\quad + \{N(V) - F(V) - (-1)^j R(V)\} \Phi(V) - \{N(U) - F(U) + R(U)\} \Phi(U) \end{aligned}$$

Corollary 2.7 ([14]). *If, in addition, $U > 2\pi$, then*

$$\sum_{U < \gamma \leq V} \Phi(\gamma) \leq \left\{ \frac{1}{2\pi} + (-1)^j q(Y) \right\} \int_U^V \Phi(y) \log \frac{y}{2\pi} dy + E_j(U, V)$$

where

$$q(y) = \frac{0.137 \log y + 0.443}{y \log y \log(y/2\pi)}$$

Define for $x \geq 1$

$$X = \sqrt{\frac{\log x}{R_0}}$$

where $R_0 = 5.69693$. Also for positive ν , positive integer m , and non-negative real T_1 and T_2 , define

$$R_m(\nu) = \{(1 + \nu)^{m+1} + 1\}^m \quad (2.8)$$

$$S_1(m, \nu) = 2 \sum_{\substack{\beta \leq 1/2 \\ 0 < \gamma \leq T_1}} \frac{2 + m\nu}{2|\rho|} \quad (2.9)$$

$$S_2(m, \nu) = 2 \sum_{\substack{\beta \leq 1/2 \\ \gamma > T_1}} \frac{R_m(\nu)}{\nu^m |\rho(\rho + 1) \cdots (\rho + m)|} \quad (2.10)$$

$$S_3(m, \nu) = 2 \sum_{\substack{\beta > 1/2 \\ 0 < \gamma \leq T_2}} \frac{(2 + m\nu) \exp(-X^2 / \log \gamma)}{2|\rho|} \quad (2.11)$$

$$S_4(m, \nu) = 2 \sum_{\substack{\beta > 1/2 \\ \gamma > T_2}} \frac{R_m(\nu) \exp(-X^2 / \log \gamma)}{\nu^m |\rho(\rho + 1) \cdots (\rho + m)|} \quad (2.12)$$

Lemma 2.8 ([14]). *Let T_1 and T_2 be non-negative real numbers. Let m be a positive integer. Let $x > 1$ and $0 < \delta < (x - 1)/(xm)$. Then*

$$\frac{1}{x} \left| \psi(x) - \left\{ x - \log 2\pi - \frac{1}{2} \log \left(1 - \frac{1}{x^2} \right) \right\} \right| \quad (2.13)$$

$$\leq \frac{1}{\sqrt{x}} \{S_1(m, \delta) + S_2(m, \delta)\} + S_3(m, \delta) + S_4(m, \delta) + \frac{m\delta}{2} \quad (2.14)$$

As

$$\frac{1}{|\rho(\rho + 1) \cdots (\rho + m)|} \leq \frac{1}{\gamma^{m+1}}$$

we can use Lemma 2.6 to write bounds for $S_j(m, \delta)$ in terms of integrals for suitable $\Phi(y)$. We note that for $m \neq 0$

$$\int_U^V y^{-(m+1)} \log \frac{y}{2\pi} dy = \frac{1 + m \log(U/2\pi)}{m^2 U^m} - \frac{1 + m \log(V/2\pi)}{m^2 V^m} \quad (2.15)$$

In the below integral let $y = \exp(zt/2m)$

$$\begin{aligned} \int_U^V y^{-(m+1)} e^{-\frac{X^2}{\log y}} \log \frac{y}{2\pi} dy &= \int_{U'}^{V'} e^{-\frac{zt}{2m}(m+1)} e^{-\frac{2mX^2}{zt}} \left\{ \frac{zt}{2m} - \log 2\pi \right\} \frac{z}{2m} e^{\frac{zt}{2m}} dt \\ &= \frac{z}{2m} \int_{U'}^{V'} e^{-\frac{zt}{2}} e^{-\frac{2mX^2}{zt}} \left\{ \frac{zt}{2m} - \log 2\pi \right\} dt \\ &= \frac{z^2}{4m^2} \int_{U'}^{V'} e^{-\frac{zt}{2} - \frac{2mX^2}{zt}} t dt \\ &\quad - \frac{z}{2m} \log 2\pi \int_{U'}^{V'} e^{-\frac{zt}{2} - \frac{2mX^2}{zt}} dt \\ &= \frac{z^2}{4m^2} \int_{U'}^{V'} e^{-\frac{z}{2}(t + \frac{4mX^2}{z^2t})} t dt \\ &\quad - \frac{z}{2m} \log 2\pi \int_{U'}^{V'} e^{-\frac{z}{2}(t + \frac{4mX^2}{z^2t})} dt, \end{aligned}$$

where $z = 2X\sqrt{m}$, $U' = (2m/z) \log U$, $V' = (2m/z) \log V$. So

$$\begin{aligned} & \frac{z^2}{4m^2} \int_{U'}^{V'} e^{-\frac{z}{2}(t + \frac{4mX^2}{z^2t})} t dt - \frac{z}{2m} \log 2\pi \int_{U'}^{V'} e^{-\frac{z}{2}(t + \frac{4mX^2}{z^2t})} dt \\ &= \frac{z^2}{2m^2} \{K_2(z, U') - K_2(z, V')\} - \frac{z}{m} \log 2\pi \{K_1(z, U') - K_1(z, V')\} \end{aligned}$$

Hence,

$$\begin{aligned} & \int_U^V y^{-(m+1)} e^{-\frac{X^2}{\log y}} \log \frac{y}{2\pi} dy \\ &= \frac{z^2}{2m^2} \{K_2(z, U') - K_2(z, V')\} - \frac{z}{m} \log 2\pi \{K_1(z, U') - K_1(z, V')\} \end{aligned} \quad (2.16)$$

Also if we let $y = \exp(X^2/t)$, we get

$$\begin{aligned} \int_U^V y^{-1} e^{-\frac{X^2}{\log y}} \log \frac{y}{2\pi} dy &= \int_{U''}^{V''} e^{-\frac{X^2}{t}} e^{-t} \left\{ \frac{X^2}{t} - \log 2\pi \right\} \left(-\frac{X^2}{t^2} e^{\frac{X^2}{t}} \right) dt \\ &= -X^4 \int_{U''}^{V''} t^{-3} e^{-t} dt + X^2 \log 2\pi \int_{U''}^{V''} t^{-2} e^{-t} dt \\ &= X^4 \{\Gamma(-2, V'') - \Gamma(-2, U'')\} \\ &\quad - X^2 \log 2\pi \{\Gamma(-1, V'') - \Gamma(-1, U'')\}, \end{aligned} \quad (2.17)$$

where $U'' = X^2/\log U$, $V'' = X^2/\log V$ and

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

is incomplete gamma function.

Theorem 2.9. *If $\log x > 110$, then*

$$|\psi(x) - x| < x\varepsilon(x),$$

$$|\vartheta(x) - x| < x\varepsilon(x),$$

where

$$\varepsilon(x) = 1.062253 \left(1 - \frac{0.900377}{2X} \right) X^{3/4} e^{-X}, \quad (2.18)$$

and

$$X = \sqrt{\frac{\log x}{R_0}}, \quad R_0 = 5.69693.$$

Proof. Take $m = 1$ and $T_1 = T_2 = 0$ in (2.8) through (2.13). By Lemma 2.2,

$$S_1(1, \delta) + S_2(1, \delta) < (0.0463) \frac{2 + 2\delta + \delta^2}{\delta}.$$

Also, as $\beta = \frac{1}{2}$ for $|\gamma| \leq A$, and the zeros off the critical line occur in pairs which are symmetrical with respect to this line, we have

$$S_3(1, \delta) + S_4(1, \delta) \leq \frac{2 + 2\delta + \delta^2}{\delta} \sum_{\gamma > A} \phi_1(\gamma),$$

where

$$\phi_m(y) = \frac{e^{-X^2/\log y}}{y^{m+1}}.$$

We appeal to Corollary 2.7 with $\Phi(y) = \phi_1(y)$, $j = 0$, $U = A$, $V = \infty$, and $W = W_1$, where for $m > -1$

$$W_m = \exp(X/\sqrt{m+1}).$$

Not that $q(Y) \leq q(A)$. Also, as $N(A) = F(A)$, we have

$$E_0 = 2R(Y)\phi_1(Y) - R(A)\phi_1(A). \quad (2.19)$$

Since $K_\nu(z, x) + K_{-\nu}(z, 1/x) = K_\nu(z)$ and (2.16), we have

$$\int_A^\infty \phi_1(y) \log \frac{y}{2\pi} dy \leq 2X \{XK_2(2X) - \log(2\pi)K_1(2X)\} \leq 2X^2K_2(2X).$$

Then, by (2.6) and (2.7), we conclude

$$\begin{aligned} \sum_{\gamma > A} \phi_1(\gamma) &\leq \left\{ \frac{1}{2\pi} + \frac{0.137 \log A + 0.443}{A \log A \log(A/2\pi)} \right\} \int_A^\infty \phi_1(y) \log \frac{y}{2\pi} dy + E_0 \\ &< \left\{ \frac{1}{2\pi} + \frac{0.137 \log A + 0.443}{A \log A \log(A/2\pi)} \right\} (2X^2) \{K_2(2X)\} + E_0 \\ &\leq \left\{ \frac{1}{2\pi} + \frac{0.137 \log A + 0.443}{A \log A \log(A/2\pi)} \right\} (2X^2) \left\{ 1 + \frac{15}{16X} + \frac{105}{512X^2} \right\} \sqrt{\frac{\pi}{4X}} e^{-2X} + E_0 \\ &< (0.28209479177389) \left\{ 1 + \frac{15}{16X} + \frac{105}{512X^2} \right\} X^{3/2} e^{-2X} + E_0. \end{aligned} \quad (2.20)$$

If $W_1 \leq A$, then $Y = A$. Then by (2.19)

$$E_0 = R(A)\phi_1(A) = \frac{R(A)}{A^2} \{e^{-X^2/\log A} X^{1/2} e^{2X}\} X^{-1/2} e^{-2X}.$$

As the expression

$$\exp\left\{-\frac{X^2}{\log A} + \frac{1}{2} \log X + 2X\right\}$$

takes its maximum at

$$X = \frac{1}{2} \log A + \frac{1}{2} \sqrt{\log^2 A + \log A}$$

we conclude that

$$E_0 < 1.53 \cdot 10^{-11} X^{-1/2} e^{-2X}. \quad (2.21)$$

If $W_1 > A$, then $Y = W_1$ and $X > 40$. As $R(y)/\log y$ is decreasing for $y > e^e$, (2.19) gives

$$\begin{aligned} E_0 &< 2R(Y)\phi_1(Y) = 2 \frac{R(Y)}{\log Y} \phi_1(Y) \log Y \\ &< 2 \frac{R(A)}{\log A} \phi_1(W_1) \log W_1 \\ &= 2 \frac{R(A)}{\log A} \frac{X}{\sqrt{2}} e^{-2\sqrt{2}X} \\ &= \sqrt{2} \frac{R(A)}{\log A} X e^{-2\sqrt{2}X} \\ &= \sqrt{2} \frac{R(A)}{\log A} X^{3/2} e^{2X-2\sqrt{2}X} X^{-1/2} e^{-2X} \\ &< (3.56 \cdot 10^{-13}) X^{-1/2} e^{-2X}. \end{aligned}$$

so that we conclude (2.21) for this case also. Then by (2.20)

$$\sum_{\gamma > A} \phi_1(\gamma) < (0.282094791774) \left\{ 1 + \frac{15}{16X} + \frac{105}{512X^2} \right\} X^{3/2} e^{-2X}. \quad (2.22)$$

As $\log x \geq 110$

$$\frac{0.0463}{\sqrt{x}} = (0.0463) e^{-\frac{1}{2} R_0 X^2} < 10^{-21} X^{-1/2} e^{-2X}. \quad (2.23)$$

Choose

$$\delta = 2(0.282094791775)^{1/2} \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X}. \quad (2.24)$$

So

$$\begin{aligned} \frac{S_1(1, \delta) + S_2(1, \delta)}{\sqrt{x}} + S_3(1, \delta) + S_4(1, \delta) &< \frac{2 + 2\delta + \delta^2}{\delta} \left(\frac{0.0463}{\sqrt{x}} + \sum_{\gamma > A} \phi_1(\gamma) \right) \\ &< \frac{2 + 2\delta + \delta^2}{\delta} \left(10^{-21} X^{-1/2} e^{-2X} + (0.282094791774) \left\{ 1 + \frac{15}{16X} + \frac{105}{512X^2} \right\} X^{3/2} e^{-2X} \right) \\ &< \frac{2 + 2\delta + \delta^2}{\delta} (0.282094791775) \left\{ 1 + \frac{15}{16X} + \frac{105}{512X^2} \right\} X^{3/2} e^{-2X} \\ &< \frac{2}{\delta} (0.282094791775) \left(1 + \frac{15}{32X} \right)^2 X^{3/2} e^{-2X}. \end{aligned}$$

Let

$$\varepsilon_1(x) = \frac{S_1(1, \delta) + S_2(1, \delta)}{\sqrt{x}} + S_3(1, \delta) + S_4(1, \delta) + \frac{\delta}{2}.$$

Then

$$\begin{aligned} \varepsilon_1(x) &< \frac{2}{\delta} (0.282094791775) \left(1 + \frac{15}{32X} \right)^2 X^{3/2} e^{-2X} + \frac{\delta}{2} \\ &= (0.282094791775)^{1/2} \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X} + (0.282094791775)^{1/2} \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X} \\ &= 2(0.282094791775)^{1/2} \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X} \\ &< (1.06225193203) \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X}. \end{aligned}$$

So

$$|\psi(x) - \{x - \log 2\pi - \frac{1}{2} \log(1 - \frac{1}{x^2})\}| < x\varepsilon(x),$$

where

$$\varepsilon(x) = (1.06225193203) \left\{ 1 + \frac{15}{32X} \right\} X^{3/4} e^{-X}.$$

By Theorem 13 of [13],

$$|\psi(x) - \vartheta(x)| < 1.43\sqrt{x}$$

Thus, it would appear that for $\vartheta(x)$ we should increase $\varepsilon(x)$ by $1.43/\sqrt{x}$. However, we can treat it as in (2.23) to show it is absorbed when we round up some of the coefficients. \square

Theorem 2.10. *If $\log x \geq 110$*

$$|\psi(x) - x| < x\varepsilon^*(x),$$

where

$$\varepsilon^*(x) = \frac{\varepsilon(x)}{\sqrt{2}} \left\{ 1 + \frac{3 \log X}{2\sqrt{\pi(4X - 3 \log X)}} + \frac{3}{2\sqrt{\pi X}} \right\}$$

and

$$r(x) = 1 + \frac{15}{32X}.$$

Proof. Take

$$\delta = \frac{1}{\sqrt{2}}(1.06225193203)\{1 + \frac{15}{32X}\}X^{3/4}e^{-X}. \quad (2.25)$$

We may assume $X \geq 33.4$ or (more accurate $x \geq e^{6344}$), since if $X \leq 33.36$ or (more accurate $x \leq e^{6343}$), $\varepsilon^*(x) > \varepsilon(x)$. We take $m = 1$, $T_1 = 0$, and

$$T_2 = X^{-3/4}e^X. \quad (2.26)$$

As $X > 32$ we have $A < T_2 < e^X = W_0$ and $W_1 < T_2$.

We can treat $\{S_1(1, \delta) + S_2(1, \delta)\}/\sqrt{x}$ and the error terms $E_j(U, V)$ arising from the use of Corollary 2.7, as we did in the proof of the previous theorem. Thus we can proceed as though

$$S_3(1, \delta) = \frac{2+\delta}{2} \sum_{0 < \gamma \leq T_2} \phi_0(\gamma) < \frac{2+\delta}{2} \left(\frac{1}{2\pi} - q(T_2) \right) \int_A^{T_2} \phi_0(y) \log \frac{y}{2\pi} dy. \quad (2.27)$$

If $\nu \leq 1$ and $x > 0$, then

$$\Gamma(\nu, x) = \int_x^\infty t^{\nu-1} e^{-t} dt \leq x^{\nu-1} \int_x^\infty e^{-t} dt = x^{\nu-1} e^{-x}$$

and

$$\Gamma(\nu, x) = \int_x^\infty t^{\nu-1} e^{-t} dt \geq (1+x)^{\nu-1} e^{-x}.$$

Hence, by (2.17), we have in effect

$$\begin{aligned} S_3(1, \delta) &< \frac{2+\delta}{4\pi} \{X^4\{\Gamma(-2, V'') - \Gamma(-2, U'')\} - X^2 \log 2\pi\{\Gamma(-1, V'') - \Gamma(-1, U'')\}\} \\ &\leq \frac{2+\delta}{4\pi} \{X^4\Gamma(-2, V'') - X^2 \log 2\pi\Gamma(-1, V'')\} \\ &< \frac{2+\delta}{4\pi} \{X^4(V'')^{-3} - X^2 \log 2\pi(1+V'')^{-2}\} e^{-V''}, \end{aligned} \quad (2.28)$$

where

$$V'' = \frac{X^2}{\log T_2} = \frac{4X^2}{4X - 3 \log X}. \quad (2.29)$$

Then

$$V'' > X + \frac{3}{4} \log X, \quad e^{-V''} < X^{-3/4} e^{-X}. \quad (2.30)$$

Also

$$\begin{aligned} X^4(V'')^{-3} - X^2 \log 2\pi(V'')^{-2} &= \left(1 - \frac{3 \log X}{4X}\right)^2 \\ &\quad \left(X - \frac{3}{4} \log X - \frac{\log 2\pi}{(1 + 1/X - 3 \log(X)/(4X^2))^2}\right) < X. \end{aligned}$$

So, effectively

$$S_3(1, \delta) < \frac{2+\delta}{4\pi} X^{1/4} e^{-X}. \quad (2.31)$$

Similarly, we can proceed as though

$$S_4(1, \delta) = \frac{2 + 2\delta + \delta^2}{\delta} \sum_{\gamma > T_2} \phi_1(\gamma) < \frac{2 + 2\delta + \delta^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \int_{T_2}^{\infty} \phi_1(y) \log \frac{y}{2\pi} dy. \quad (2.32)$$

By (2.16)

$$\int_{T_2}^{\infty} \phi_1(y) \log \frac{y}{2\pi} dy = 2\{X^2 K_2(2X, U') - X \log(2\pi) K_1(2X, U')\} \leq 2X^2 K_2(2X, U'), \quad (2.33)$$

where

$$U' = \frac{1}{X} \log T_2 = 1 - \frac{3 \log X}{4X}. \quad (2.34)$$

Write temporarily

$$q = \frac{3 \log X}{4X}, \quad y = \frac{1}{\sqrt{2}} \left(\sqrt{U'} - \frac{1}{\sqrt{U'}} \right). \quad (2.35)$$

Then y is negative, and

$$y^2 = \frac{1}{2} \left(U' + \frac{1}{U'} \right) - 1 = \frac{q^2}{2(1-q)}.$$

So, by splitting the integral in Lemma 2.5 at $w = 0$, we get

$$\begin{aligned} \sqrt{2} \int_y^{\infty} e^{-2Xw^2} dw &= \sqrt{2} \int_y^0 e^{-2Xw^2} dw + \sqrt{2} \int_0^{\infty} e^{-2Xw^2} dw \\ &= \sqrt{2} \int_y^0 e^{-2Xw^2} dw + \frac{\sqrt{\pi}}{2\sqrt{X}} = \sqrt{2} \int_0^y e^{-2Xw^2} dw + \frac{\sqrt{\pi}}{2\sqrt{X}} \\ &< \sqrt{2} \int_0^y 1 dw + \frac{\sqrt{\pi}}{2\sqrt{X}} = \frac{q}{\sqrt{1-q}} + \frac{\sqrt{\pi}}{2\sqrt{X}}. \end{aligned} \quad (2.36)$$

Hence, by (2.4) we get

$$\begin{aligned} X \log(2\pi) K_1(2X, U') &< X \log(2\pi) \frac{e^{-2X}}{4X} \left\{ \left(1 + \frac{3\sqrt{2}}{8} y \right) e^{-2Xy^2} + \left(\frac{3}{8} + 2X \right) \left(\frac{\sqrt{\pi}}{2\sqrt{X}} + \frac{q}{\sqrt{1-q}} \right) \right\} \\ &\leq \frac{1}{4} \log(2\pi) e^{-2X} \left\{ 1 + \left(\frac{3}{8} + 2X \right) \left(\frac{\sqrt{\pi}}{2\sqrt{X}} + \frac{q}{\sqrt{1-q}} \right) \right\} \\ &= \frac{\sqrt{\pi}}{4} \log(2\pi) X^{3/2} e^{-2X} \left\{ \frac{1}{\sqrt{\pi} X^{3/2}} + \left(\frac{3}{16X^2} + \frac{1}{X} \right) \left(1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} \right) \right\} \end{aligned} \quad (2.37)$$

As $1 + zy^2 < e^{zy^2}$, we have $(2y^2 + 2/z)e^{-zy^2} < 2/z = 1/X$. Hence, by (2.5) we get

$$\begin{aligned} X^2 K_2(2X, U') &< X^2 \frac{e^{-2X}}{4X} \left\{ \left[\frac{35\sqrt{2}}{64} y^3 + 2y^2 + \left(\frac{105}{128z} + \frac{15}{8} \right) \sqrt{2}y + 2 + \frac{2}{z} \right] e^{-zy^2} \right. \\ &\quad \left. + \left(\frac{105}{256X} + \frac{15}{8} + 2X \right) \left(\frac{\sqrt{\pi}}{2\sqrt{X}} + \frac{q}{\sqrt{1-q}} \right) \right\} \\ &< \frac{1}{4} X e^{-2X} \left\{ 2 + \frac{1}{X} + \left(\frac{105}{256X} + \frac{15}{8} + 2X \right) \left(\frac{\sqrt{\pi}}{2\sqrt{X}} + \frac{q}{\sqrt{1-q}} \right) \right\} \\ &= \frac{1}{4} \sqrt{\pi} X^{3/2} e^{-2X} \left\{ \frac{2}{\sqrt{\pi} X} + \frac{1}{\sqrt{\pi} X^{3/2}} + \left(\frac{105}{512X^2} + \frac{15}{16X} + 1 \right) \left(1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} \right) \right\}. \end{aligned} \quad (2.38)$$

Combining with (2.33) and (2.37) gives

$$\int_{T_2}^{\infty} \phi_1(y) \log \frac{y}{2\pi} dy < \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} Q_1,$$

where

$$Q_1 = \frac{2}{\sqrt{\pi X}} + \frac{1}{\sqrt{\pi} X^{3/2}} + \left(\frac{105}{512 X^2} + \frac{15}{16 X} + 1 \right) \left(1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} \right).$$

So finally by (2.32) and (2.25)

$$\begin{aligned} S_4(1, \delta) &< \frac{2+2\delta+\delta^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \int_{T_2}^{\infty} \phi_1(y) \log \frac{y}{2\pi} dy \\ &< \frac{2+2\delta+\delta^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} Q_1 \\ &= \frac{2+2\delta+\delta^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} 2r(x)^2 \\ &\quad \times \left\{ \frac{1}{r(x)^2} \left(\frac{1}{\sqrt{\pi X}} + \frac{1}{2\sqrt{\pi} X^{3/2}} \right) \right. \\ &\quad \left. + \frac{1}{2r(x)^2} \left(\frac{105}{512 X^2} + \frac{15}{16 X} + 1 \right) \left(1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} \right) \right\} \\ &< 2 \frac{r(x)^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} \\ &\quad \times \left\{ \frac{2+2\delta+\delta^2}{r(x)^2} \left(\frac{1}{\sqrt{\pi X}} + \frac{1}{2\sqrt{\pi} X^{3/2}} \right) + \left(1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} \right) \right\} \\ &= 2 \frac{r(x)^2}{\delta} \left(\frac{1}{2\pi} + q(T_2) \right) \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} Q_2 \\ &= 2 \frac{r(x)^2}{\delta^2} \delta \left(\frac{1}{2\pi} + q(T_2) \right) \frac{\sqrt{\pi}}{2} X^{3/2} e^{-2X} Q_2 \\ &< \left(\frac{1}{2} \right) \delta Q_2, \quad (X \geq 28), \end{aligned} \tag{2.39}$$

where

$$Q_2 = 1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{2+2\delta+\delta^2}{r(x)^2} \left(\frac{1}{\sqrt{\pi X}} + \frac{1}{2\sqrt{\pi} X^{3/2}} \right). \tag{2.40}$$

So

$$\begin{aligned}
& \frac{S_1(1, \delta) + S_2(1, \delta)}{\sqrt{x}} + S_3(1, \delta) + S_4(1, \delta) + \frac{\delta}{2} \\
& < \frac{2 + 2\delta + \delta^2}{\delta} \frac{0.0463}{\sqrt{x}} + \frac{2 + \delta}{4\pi} X^{1/4} e^{-X} \\
& \quad + \frac{\delta}{2} \left\{ 1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{2 + 2\delta + \delta^2}{r(x)^2} \left(\frac{1}{\sqrt{\pi X}} + \frac{1}{2\sqrt{\pi X^{3/2}}} \right) \right\} + \frac{\delta}{2} \\
& < \frac{\delta}{2} \left\{ 1 + \frac{2q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{2 + 2\delta + \delta^2}{r(x)^2} \left(\frac{1}{\sqrt{\pi X}} + \frac{1}{2\sqrt{\pi X}} \right) \right\} + \frac{\delta}{2}, \quad (X \geq 6) \\
& = \delta \left\{ 1 + \frac{q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{2 + 2\delta + \delta^2}{2r(x)^2} \left(\frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} \right) \frac{1}{\sqrt{X}} \right\} \\
& < \delta \left\{ 1 + \frac{q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{3}{2\sqrt{\pi X}} \right\} \\
& = \frac{\varepsilon(x)}{\sqrt{2}} \left\{ 1 + \frac{q\sqrt{X}}{\sqrt{\pi(1-q)}} + \frac{3}{2\sqrt{\pi X}} \right\} \\
& = \frac{\varepsilon(x)}{\sqrt{2}} \left\{ 1 + \frac{3 \log X}{2\sqrt{\pi(4X - 3 \log X)}} + \frac{3}{2\sqrt{\pi X}} \right\}.
\end{aligned}$$

□

2.3 Numerical Bounds for $\psi(x) - x$ for Moderate Values of x

In our main table (at the end of the thesis) we tabulate values of ε against b . These have been determined so that if $x \geq e^b$, then

$$|\psi(x) - x| < \varepsilon x. \quad (2.41)$$

Let $T_2 = 0$ and

$$T_1 = \frac{1}{\delta} \left(\frac{2R_m(\delta)}{2 + m\delta} \right)^{1/m}. \quad (2.42)$$

We chose also

$$D = 963.5670402. \quad (2.43)$$

The zeros for which $0 < \gamma \leq D$ are exactly 620 in number.

$$S \equiv \sum_{0 < \gamma \leq D} \frac{1}{|\rho|} = \sum_{0 < \gamma \leq D} \frac{1}{(\gamma^2 + 1/4)^{1/2}} < 2. \quad (2.44)$$

Lemma 2.11. *With T_1 and D given by (2.42) and (2.43), if $T_1 \geq D$, $\delta > 0$, and m is a positive integer, then*

$$S_1(m, \delta) + S_2(m, \delta) < \frac{2 + m\delta}{4\pi} \left\{ \left(\log \frac{T_1}{2\pi} + \frac{1}{m} \right)^2 + \frac{1}{m^2} - 0.1580304 - 2.531837599 \frac{m}{(m+1)T_1} \right\}. \quad (2.45)$$

Proof. By (2.9) and (2.44)

$$S_1(m, \delta) = (2 + m\delta) \sum_{\substack{\beta \leq 1/2 \\ 0 < \gamma \leq T_1}} \frac{1}{|\rho|} < (2 + m\delta) \left\{ S + \sum_{D < \gamma \leq T_1} \frac{1}{\gamma} \right\}.$$

Taking $\Phi(y) = y^{-1}$, $j = 0$, $U = D$, $V = T_1$, and $W = 0$ in Lemma 2.6 gives

$$\begin{aligned} \sum_{D < \gamma \leq T_1} \frac{1}{\gamma} &\leq \frac{1}{2\pi} \int_D^{T_1} \frac{1}{y} \log \frac{y}{2\pi} dy + \left\{ 0.137 + \frac{0.443}{\log D} \right\} \int_D^{T_1} \frac{1}{y^2} dy + E_0 \\ &\leq \frac{1}{4\pi} \left\{ \log^2 \frac{T_1}{2\pi} - \log^2 \frac{D}{2\pi} \right\} + \left(0.137 + \frac{0.443}{\log D} \right) \left\{ -\frac{1}{T_1} + \frac{1}{D} \right\} + E_0. \end{aligned}$$

Then (2.44) together with $N(D) = 620$ gives

$$\begin{aligned} \frac{S_1(m, \delta)}{2 + m\delta} &< S + \frac{1}{4\pi} \left\{ \log^2 \frac{T_1}{2\pi} - \log^2 \frac{D}{2\pi} \right\} - \left(0.137 + \frac{0.443}{\log D} \right) \left\{ \frac{1}{T_1} - \frac{1}{D} \right\} \\ &\quad + \frac{1}{T_1} (N(T_1) - F(T_1) - R(T_1)) - \frac{1}{D} (N(D) - F(D) - R(D)) \\ &< 2 - \frac{1}{4\pi} \log^2 \frac{D}{2\pi} + \frac{1}{D} \left(0.137 + \frac{0.443}{\log D} - N(D) + F(D) + R(D) \right) \\ &\quad + \frac{1}{4\pi} \log^2 \frac{T_1}{2\pi} - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right) \\ &< -0.01257566 + \frac{1}{4\pi} \log^2 \frac{T_1}{2\pi} - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right). \end{aligned} \tag{2.46}$$

Since

$$S_2(m, \delta) = 2 \sum_{\substack{\beta \leq 1/2 \\ \gamma > T_1}} \frac{R_m(\delta)}{\delta^m |\rho(\rho+1) \cdots (\rho+m)|}.$$

Taking $\Phi(y) = y^{-(m+1)}$, $j = 0$, $U = T_1$, $V = \infty$, and $W = 0$ in Lemma 2.6 and using (2.15) gives

$$\begin{aligned} \frac{\delta^m S_2(m, \delta)}{2R_m(\delta)} &= \sum_{\substack{\beta \leq 1/2 \\ \gamma > T_1}} \frac{1}{|\rho(\rho+1) \cdots (\rho+m)|} < \sum_{\gamma > T_1} \frac{1}{\gamma^{m+1}} \\ &\leq \frac{1}{2\pi} \int_{T_1}^{\infty} y^{-(m+1)} \log \frac{y}{2\pi} dy + \left(0.137 + \frac{0.443}{\log T_1} \right) \int_{T_1}^{\infty} \frac{y^{-(m+1)}}{y} dy + E_0^* \\ &= \frac{1}{2\pi} \left(\frac{1}{m^2 T_1^m} + \frac{1}{m T_1^m} \log \frac{T_1}{2\pi} \right) + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1) T_1^{m+1}} + E_0^* \\ &= \frac{1}{T_1^m} \left\{ \frac{1}{2\pi m} \log \frac{T_1}{2\pi} + \frac{1}{2\pi m^2} + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1) T_1} \right\} + E_0^* \\ &= \frac{1}{T_1^m} \left\{ \frac{1}{2\pi m} \log \frac{T_1}{2\pi} + \frac{1}{2\pi m^2} + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1) T_1} + E_0^* T_1^m \right\}. \end{aligned}$$

Using (2.42) and combining with (2.46) gives

$$\begin{aligned}
& S_1(m, \delta) + S_2(m, \delta) \\
& < (2 + m\delta) \left[-0.01257566 + \frac{1}{4\pi} \log^2 \frac{T_1}{2\pi} - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right) \right] \\
& \quad + \frac{2R_m(\delta)}{\delta^m} \frac{1}{T_1^m} \left[\frac{1}{2\pi m} \log \frac{T_1}{2\pi} + \frac{1}{2\pi m^2} + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1)T_1} + E_0^* T_1^m \right] \\
& = (2 + m\delta) \left[-0.01257566 + \frac{1}{4\pi} \log^2 \frac{T_1}{2\pi} - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right) \right] \\
& \quad + (2 + m\delta) \left[\frac{1}{2\pi m} \log \frac{T_1}{2\pi} + \frac{1}{2\pi m^2} + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1)T_1} + E_0^* T_1^m \right] \\
& = \frac{2 + m\delta}{4\pi} \left[\log^2 \frac{T_1}{2\pi} + 4\pi \left\{ -0.01257566 - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right) \right\} \right] \\
& \quad + \frac{2 + m\delta}{4\pi} \left[\frac{2}{m} \log \frac{T_1}{2\pi} + \frac{2}{m^2} + 4\pi \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1)T_1} + 4\pi E_0^* T_1^m \right] \\
& = \frac{2 + m\delta}{4\pi} \left\{ \left(\log \frac{T_1}{2\pi} + \frac{1}{m} \right)^2 + \frac{1}{m^2} + J \right\},
\end{aligned}$$

where

$$\begin{aligned}
J & = 4\pi \left\{ -0.01257566 - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} - N(T_1) + F(T_1) + R(T_1) \right) \right\} \\
& \quad + 4\pi \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1)T_1} + 4\pi E_0^* T_1^m \\
& = 4\pi \left\{ -0.01257566 - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} \right) + \left(0.137 + \frac{0.443}{\log T_1} \right) \frac{1}{(m+1)T_1} \right\} \\
& < 4\pi \left\{ -0.01257566 - \frac{1}{T_1} \left(0.137 + \frac{0.443}{\log D} \right) \left(1 - \frac{1}{m+1} \right) \right\} \\
& < -0.1580304 - 2.531837599 \frac{m}{(m+1)T_1}.
\end{aligned}$$

□

Theorem 2.12. Let $T_1 \geq D$. Let m be a positive integer, let Ω_1 denote the right side of (2.45) and let

$$\begin{aligned}
\Omega_2 & = (0.159155) \frac{R_m(\delta)z}{2m^2\delta^m} \{ zK_2(z, A') - 2m \log(2\pi) K_1(z, A') \} \\
& \quad + \frac{R_m(\delta)}{\delta^m} \{ 2R(Y)\phi_m(Y) - R(A)\phi_m(A) \},
\end{aligned} \tag{2.47}$$

where $z = 2X\sqrt{m} = 2\sqrt{mb/R_0}$, $A' = (2m/z) \log A$, $Y = \max\{A, \exp \sqrt{b/(m+1)R_0}\}$. If $b > 1/2$ and $0 < \delta < (1 - e^{-b})/m$, then

$$|\psi(x) - x| < \varepsilon x, \quad (x \geq e^b),$$

where

$$\varepsilon = \Omega_1 e^{-b/2} + \Omega_2 + \frac{m\delta}{2} + e^{-b} \log 2\pi.$$

Proof. Take $T_2 = 0$, then $S_3(m, \delta) = 0$ and by Corollary 2.7 and 2.16,

$$\begin{aligned} \frac{\delta^m}{R_m(\delta)} S_4(m, \delta) &= \sum_{\substack{\beta > 1/2 \\ \gamma > 0}} \frac{e^{-X^2/\log \gamma}}{|\rho(\rho+1) \cdots (\rho+m)|} < \sum_{\gamma > A} \frac{e^{-X^2/\log \gamma}}{\gamma^{m+1}} = \sum_{\gamma > A} \phi_m(\gamma) \\ &\leq \{1 + q(Y)\} \int_A^\infty \phi_m(y) \log \frac{y}{2\pi} dy + E_0 \\ &< 0.159155 \left(\frac{z^2}{2m^2} K_2(z, A') - \frac{z}{m} \log(2\pi) K_1(z, A') \right) \\ &\quad + \{2R(Y)\phi_m(Y) - R(A)\phi_m(A)\}. \end{aligned}$$

So

$$S_3(m, \delta) + S_4(m, \delta) < \Omega_2.$$

Since

$$\frac{1}{x} |\psi(x) - x + \log 2\pi + \frac{1}{2} \log(1 - \frac{1}{x^2})| < \frac{S_1(m, \delta) + S_2(m, \delta)}{\sqrt{x}} + S_3(m, \delta) + S_4(m, \delta) + \frac{m\delta}{2}.$$

So

$$\begin{aligned} \frac{1}{x} |\psi(x) - x| &< \frac{S_1(m, \delta) + S_2(m, \delta)}{\sqrt{x}} + S_3(m, \delta) + S_4(m, \delta) + \frac{m\delta}{2} + \frac{\log 2\pi}{x} \\ &< \frac{\Omega_1}{\sqrt{x}} + \Omega_2 + \frac{m\delta}{2} + \frac{\log 2\pi}{x} \\ &\leq \Omega_1 e^{-b/2} + \Omega_2 + \frac{m\delta}{2} + e^{-b} \log 2\pi. \end{aligned}$$

□

Theorem 2.13. Let $T_1 \geq D$ and $A \leq T_2 \leq \exp \sqrt{b/R}$. Let m be a positive integer and let

$$\begin{aligned} \Omega_3 &= \frac{2+m\delta}{4\pi} [X^4 \{\Gamma(-2, T'') - \Gamma(-2, A'')\} - X^2 \log(2\pi) \{\Gamma(-1, T'') - \Gamma(-1, A'')\}] \\ &\quad + \frac{2+m\delta}{2} \{2R(T_2)\phi_0(T_2) - R(A)\phi_0(A)\} + \Omega_2^*, \end{aligned} \tag{2.48}$$

where $A'' = b/(R_0 \log A)$, $T'' = b/(R_0 \log T_2)$, and Ω_2^* is obtained from Ω_2 by deleting the term $-R(A)\phi_m(A)$ in (2.47) and then replacing A by T_2 in the definition of A' and Y . If $b > 1/2$ and $0 < \delta < (1 - e^{-b})/m$, then (2.41) holds for all $x \geq e^b$, where

$$\varepsilon = \Omega_1 e^{-b/2} + \Omega_3 + \frac{m\delta}{2} + e^{-b} \log 2\pi. \tag{2.49}$$

If we use the following bounds for $\Gamma(\nu, x)$ and $\nu < 1$, for large b we get a better bounds than those given in three theorems before the last one;

$$\frac{x^\nu e^{-x}}{x+1-\nu} < \Gamma(\nu, x) < x^{\nu-3} e^{-x} \{x^2 + (\nu-1)x + (\nu-1)(\nu-2)\}, \quad (x > 0, \nu < 1).$$

2.4 Bounds for $\vartheta(x) - x$ for Large Values of x

Theorem 2.14 ([14]). *We have*

$$\begin{aligned} \vartheta(x) &< 1.001, 102x, & (x > 0), \\ 0.998, 684x &< \vartheta(x), & (x \geq 1, 319, 007), \\ \psi(x) - \vartheta(x) &< 1.001, 102\sqrt{x} + 3\sqrt[3]{x}, & (x > 0), \\ 0.998, 684\sqrt{x} &< \psi(x) - \vartheta(x), & (x \geq 121). \end{aligned}$$

Corollary 2.15 ([14]). *We have*

$$\begin{aligned} \vartheta(x) &> 0.998x, & (x \geq 487, 381), \\ \vartheta(x) &> 0.995x, & (x \geq 89, 387), \\ \vartheta(x) &> 0.990x, & (x \geq 32, 057), \\ \vartheta(x) &> 0.985x, & (x \geq 11, 927). \end{aligned}$$

Theorem 2.16 ([14]). *If $x \geq 10^8$, then*

$$\begin{aligned} |\psi(x) - x| &< 0.0242269 \frac{x}{\log x}, \\ |\vartheta(x) - x| &< 0.0242269 \frac{x}{\log x}. \end{aligned}$$

Corollary 2.17 ([14]). *If $x \geq 525, 752$, then*

$$\vartheta(x) - x \leq \psi(x) - x < 0.024, 2334 \frac{x}{\log x}.$$

Corollary 2.18 ([14]). *We have*

$$\begin{aligned} |\vartheta(x) - x| &< 0.024, 2334 \frac{x}{\log x}, & (x \geq 758, 699), \\ |\vartheta(x) - x| &< \frac{1}{40} \frac{x}{\log x}, & (x \geq 678, 407). \end{aligned}$$

Theorem 2.19 ([14]). *If $x > 1$, then*

$$\begin{aligned} |\psi(x) - x| &< \eta_k \frac{x}{\log^k x}, \\ |\vartheta(x) - x| &< \eta_k \frac{x}{\log^k x}, \end{aligned}$$

where

$$\eta_2 = 8.6853, \quad \eta_3 = 11, 762, \quad \eta_4 = 1.8559 \cdot 10^7.$$

Theorem 2.20. *If $\varepsilon(x)$ is defined as (2.18), then*

$$\begin{aligned}\vartheta(x) - x &\leq \psi(x) - x < x\varepsilon(x), & (x > 0), \\ \psi(x) - x &\geq \vartheta(x) - x > -x\varepsilon(x), & (x \geq 39.4).\end{aligned}$$

2.5 Improved estimates for $\psi - \vartheta$

In this section we give some results from [3] to approximate the difference $\psi - \vartheta$ in terms of ψ in quite a simple form. As consequences we deduce some estimates for $\psi - \vartheta$.

Theorem 2.21 ([3]). *For every $x > 0$ we have*

$$\begin{aligned}\psi(x) - \vartheta(x) &\leq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}), \\ \psi(x) - \vartheta(x) &\geq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/7}).\end{aligned}$$

Theorem 2.22 ([3]). *We have*

$$\begin{aligned}\psi(x) - \vartheta(x) &< \sqrt{x} + \frac{4}{3}\sqrt[3]{x}, & (0 < x \leq 10^8), \\ \psi(x) - \vartheta(x) &> \sqrt{x} + \frac{2}{3}\sqrt[3]{x}, & (2187 \leq x \leq 10^8).\end{aligned}$$

Theorem 2.23 ([3]).

$$\psi(x) < x + 0.656\sqrt{x} + \frac{4}{3}\sqrt[3]{x}, \quad (0 < x \leq 10^8).$$

For $1427 \leq x \leq 3298$, $3299 \leq x \leq 19371$ or $19373 \leq x \leq 10^8$

$$\psi(x) > x - 0.833\sqrt{x} + \frac{2}{3}\sqrt[3]{x}.$$

Theorem 2.24 ([3]).

$$\begin{aligned}\psi(x) - \vartheta(x) &< \sqrt{x} + \frac{6}{5}\sqrt[3]{x}, & (10^8 \leq x \leq 10^{16}), \\ \psi(x) - \vartheta(x) &> \sqrt{x} + \frac{6}{7}\sqrt[3]{x}, & (10^8 \leq x \leq 10^{16}).\end{aligned}$$

Theorem 2.25 ([3]). *With the aid of a computer, it can be easily verified that*

$$\begin{aligned}\psi(x) - \vartheta(x) &< \sqrt{x} + \frac{6}{5}\sqrt[3]{x}, & (8,236,167 \leq x \leq 10^{16}), \\ \psi(x) - \vartheta(x) &> \sqrt{x} + \frac{6}{7}\sqrt[3]{x}, & (2,036,329 \leq x \leq 10^{16}).\end{aligned}$$

The results which have given so far are strictly elementary. However, in order to estimate $\psi(x) - \vartheta(x)$ for $x > 10^{16}$, one needs the following bounds for ψ which were deduced by Schoenfeld

[15], using powerful analytical methods.

$$\begin{aligned}
|\psi(x) - x| &< 0.00119721x, & (10^8 \leq x < e^{18.43}), \\
|\psi(x) - x| &< 0.0011930x, & (e^{18.43} \leq x < e^{18.44}), \\
|\psi(x) - x| &< 0.0011885x, & (e^{18.44} \leq x < e^{18.45}), \\
|\psi(x) - x| &< 0.0011839x, & (e^{18.45} \leq x < e^{18.46}), \\
|\psi(x) - x| &< 0.0011615x, & (e^{18.46} \leq x < e^{18.47}), \\
|\psi(x) - x| &< 0.0010765x, & (e^{18.7} \leq x < e^{19}), \\
|\psi(x) - x| &< 0.00096161x, & (x \geq e^{19}).
\end{aligned}$$

Theorem 2.26 ([3]).

$$\begin{aligned}
\psi(x) - \vartheta(x) &< 1.001\sqrt{x} + 1.1\sqrt[3]{x}, & (x \geq 10^{16}), \\
\psi(x) - \vartheta(x) &< 1.001\sqrt{x} + \sqrt[3]{x}, & (x \geq e^{38}), \\
\psi(x) - \vartheta(x) &> 0.999\sqrt{x} + 0.9\sqrt[3]{x}, & (x \geq 10^{16}), \\
\psi(x) - \vartheta(x) &> 0.999\sqrt{x} + \sqrt[3]{x}, & (x \geq e^{38}).
\end{aligned}$$

For $x = 10^8$, we have $\delta = 2.44 \cdot 10^{-4}$, $m = 2$, $\varepsilon = 0.00118294$. For $x = 10^{16}$, we have $\delta = 5.24 \cdot 10^{-8}$, $m = 2$, $\varepsilon = 4.66629 \cdot 10^{-7}$.

Theorem 2.27. *We have*

$$\begin{aligned}
\vartheta(x) &< 1.000027651x, & (x > 0), \\
\vartheta(x) &> 0.99871149x, & (x \geq 10^8).
\end{aligned}$$

Proof. If $8 \cdot 10^{11} \leq x < e^{28}$, then

$$\begin{aligned}
\vartheta(x) &< \psi(x) - \sqrt{x} - \frac{6}{7}\sqrt[3]{x} < \left\{ 1.0000284888 - \frac{1}{\sqrt{x}} - \frac{6}{7} \frac{1}{\sqrt[3]{x^2}} \right\} x \\
&< \left\{ 1.0000284888 - e^{-28/2} - \frac{6}{7} e^{-28(2/3)} \right\} x \\
&< 1.000027651x.
\end{aligned}$$

By handling the intervals $[e^{28}, e^{29})$, etc., similarly, we derive the same inequality. And for $x \geq e^{28}$ we use the table and $\vartheta(x) < \psi(x)$. This proves for all $x \geq 8 \cdot 10^{11}$. For $x < 8 \cdot 10^{11}$, it follows from (4.5) of [13] and Dusart[4] which says $\vartheta(x) < x$ in this domain.

If $8 \cdot 10^{11} \leq x < 10^{16}$, then

$$\begin{aligned}
\vartheta(x) &> \psi(x) - \sqrt{x} - \frac{6}{5}\sqrt[3]{x} > \left\{ 1 - 0.0000284888 - \frac{1}{\sqrt{x}} - \frac{6}{5} \frac{1}{\sqrt[3]{x^2}} \right\} x \\
&> \left\{ 0.9999715112 - (8 \cdot 10^{11})^{-1/2} - \frac{6}{5} (8 \cdot 10^{11})^{-(2/3)} \right\} x \\
&> 0.9999703792x.
\end{aligned}$$

If $x \geq 10^{16}$

$$\begin{aligned}\vartheta(x) &> \psi(x) - 1.001\sqrt{x} - 1.1\sqrt[3]{x} > \left\{1 - 4.66629 \cdot 10^{-7} - 1.001\frac{1}{\sqrt{x}} - 1.1\frac{1}{\sqrt[3]{x^2}}\right\}x \\ &> \left\{0.999999533371 - (1.001)10^{-8} - (1.1)10^{-16(2/3)}\right\}x \\ &> 0.9999995233373x\end{aligned}$$

□

Theorem 2.28. If $x \geq 8 \cdot 10^{11}$

$$\begin{aligned}|\psi(x) - x| &< 0.000797686 \frac{x}{\log x}, \\ |\vartheta(x) - x| &< 0.000821232 \frac{x}{\log x}.\end{aligned}$$

Proof. If $8 \cdot 10^{11} \leq x < e^{28}$, then

$$\begin{aligned}\vartheta(x) - x &> \psi(x) - x - \sqrt{x} - \frac{6}{5}\sqrt[3]{x} \\ &> - \left\{ \left(0.0000284888 + \frac{1}{\sqrt{x}} + \frac{6}{5} \frac{1}{\sqrt[3]{x^2}} \right) \log x \right\} \frac{x}{\log x} \\ &> - \left\{ \left(0.0000284888 + e^{-28/2} + \frac{6}{5} e^{-28(2/3)} \right) (28) \right\} \frac{x}{\log x} \\ &> - 0.000821232 \frac{x}{\log x}.\end{aligned}$$

We continue to use the table in this way until e^{35} . If $10^{16} \leq x < e^{40}$

$$\begin{aligned}\vartheta(x) - x &> \psi(x) - x - 1.001\sqrt{x} - 1.1\sqrt[3]{x} \\ &> - \left\{ \left(4.66629 \cdot 10^{-7} + 1.001\frac{1}{\sqrt{x}} + 1.1\frac{1}{\sqrt[3]{x^2}} \right) \right\} \frac{x}{\log x} \\ &> - \left\{ \left(4.66629 \cdot 10^{-7} + (1.001)e^{-20} + (1.1)e^{-40(2/3)} \right) 40 \right\} x \\ &> - 0.00001874781 \frac{x}{\log x}.\end{aligned}$$

We continue again until e^{1000} . For $x \geq e^{1000}$, we apply Theorem 2.9 and note that $\varepsilon(x) \log x < 0.012559$, so that

$$\vartheta(x) - x > -\{\varepsilon(x) \log x\} \frac{x}{\log x} > -0.012559 \frac{x}{\log x}.$$

□

Theorem 2.29. If $x > 8 \cdot 10^{11}$, then

$$|\psi(x) - x| < \eta_k \frac{x}{\log^k x},$$

where

Proof. If $x \geq 8 \cdot 10^{11}$ we proceed as previous theorem. For $x \geq 8 \cdot 10^{11}$ from table we get.

If $1 < x < 8 \cdot 10^{11}$, since $\vartheta(x) < x$, we have

$$\begin{aligned}\psi(x) - x &< \vartheta(x) - x + \sqrt{x} + \frac{4}{3}\sqrt[3]{x} < \left\{ \frac{\log^k x}{\sqrt{x}} + \frac{4}{3} \frac{\log^k x}{\sqrt[3]{x^2}} \right\} \frac{x}{\log^k x} \\ &\leq \left\{ \frac{(2k)^k}{e^k} + \frac{4}{3} \frac{(3k/2)^k}{e^k} \right\} \frac{x}{\log^k x}, \quad (k = 1, 2, 3, 4)\end{aligned}$$

Table 1: $x > 8 \cdot 10^{11}$

k	1	2	3	4
η_k	0.000797686	0.0223352	0.625386	1230

For $k = 1, 2, 3, 4$; and

$$\begin{aligned}
\psi(x) - x &> \vartheta(x) - x + \sqrt{x} + \frac{2}{3}\sqrt[3]{x} > -2.06\sqrt{x} + \sqrt{x} + \frac{2}{3}\sqrt[3]{x} \\
&= \left\{ -1.06 \frac{\log^k x}{\sqrt{x}} + \frac{2}{3} \frac{\log^k x}{\sqrt[3]{x^2}} \right\} \frac{x}{\log^k x} \\
&> -c_k \frac{x}{\log^k x}
\end{aligned}$$

where $c_1 = 0.445$ and $c_2 = 1.592$ and $c_3 = 8.887$ and $c_4 = 66.8894$. □

Theorem 2.30. For $x \geq 8 \cdot 10^{11}$

$$|\vartheta(x) - x| < \eta_k \frac{x}{\log^k x}$$

where

Table 2: η

k	1	2	3	4
η_k	0.000821232	0.0229945	0.643846	1230

Proof. For $1 < x < 10^8$

$$\begin{aligned}
\vartheta(x) - x &> -2.06\sqrt{x} = -2.06 \frac{\log^k x}{\sqrt{x}} \cdot \frac{x}{\log^k x} \\
&\geq -2.06 \frac{(2k)^k}{e^k} \frac{x}{\log^k x}
\end{aligned}$$

for $k = 1, 2, 3, 4$. □

Theorem 2.31. If $\varepsilon(x)$ is defined as in Theorem 2.9, then

$$\vartheta(x) - x \leq \psi(x) - x < x\varepsilon(x), \quad (x > 0)$$

$$\psi(x) - x \geq \vartheta(x) - x > -x\varepsilon(x), \quad (x \geq 71)$$

Proof. We need to verify them for $x < e^{110}$. As $\varepsilon(x)$ increases for $1 < x < 12644$ and decreases for $x > 12645$. We have $0.0357 < \varepsilon(x) < 0.2304221$ for $5 \leq x < e^{110}$. From the table we deduce them for $10^8 \leq x < e^{110}$. For $132 \leq x < 10^8$, we have $\varepsilon(x) > 0.204$. Hence by (3.35) of Rosser 1961,

$$\psi(x) < 1.04x < (1 + \varepsilon(x))x$$

For $1 < x < 132$, we use direct computation.

From Rosser 1961, For $110 \leq x < 10^8$

$$\vartheta(x) > 0.84x > (1 - \varepsilon(x))x$$

For $71 \leq x < 110$, we use direct computation.

Second method. By Theorem 9 of [14]

$$\psi(x) - x < x\varepsilon_3(x), \quad (x > 0)$$

where ε_3 is defined in (3.9) of [14]. On the other hand $\varepsilon_3(x) < \varepsilon(x)$ for $408 < x < e^{190}$. By computation for smaller values. The same hold for $\vartheta(x) - x > -x\varepsilon(x)$. \square

2.6 Sharper bounds for $|\psi(x) - x|$ and $|\vartheta(x) - x|$

Lemma 2.32 ([14]). *If $\nu \leq 1$, $z > 0$, and $x > 1$, we have*

$$K_\nu(z, x) < Q_\nu(z, x)$$

where

$$Q_\nu(z, x) = \frac{x^{\nu+1}}{z(x^2 - 1)} H^z(x), \quad H(t) = e^{-\frac{1}{2}(t+1/t)}$$

Lemma 2.33 ([14]). *If $z > 0$ and $x > 0$, then*

$$(x-1)Q_1(z, x) + (1 + \frac{2}{z} - \frac{2}{z(1+x)^2})K_1(z, x) < K_2(z, x) < (x-1)Q_1(z, x) + (1 + \frac{2}{z})K_1(z, x)$$

Corollary 2.34 ([14]). *If $z > 0$ and $x > 1$, then*

$$K_2(z, x) < (x + \frac{2}{z})Q_1(z, x)$$

Theorem 2.35. *Let*

$$\varepsilon_0(x) = \sqrt{8/\pi} X^{1/2} e^{-X}$$

Then

$$|\psi(x) - x| < x\varepsilon_0(x), \quad (x \geq 3)$$

and

$$|\vartheta(x) - x| < x\varepsilon_0(x), \quad (x \geq 3)$$

Proof. The main part of the proof is concerned with large x in which case the proof is similar to Theorem 2.10, but we ultimately take $m = 2$ rather than $m = 1$. In place of (2.26), we let

$$T_2 = e^{\nu x}, \tag{2.50}$$

where ν will be specified later. We assume that ν, m, X are such that

$$T_2 \geq A, \quad \frac{1}{\sqrt{m+1}} \leq \nu \leq 1 \tag{2.51}$$

from which we deduce

$$X \geq \nu X = \log T_2 \geq \log A$$

and

$$W_m = e^{X/\sqrt{m+1}} \leq T_2 = e^{\nu X} \leq e^X = W_0.$$

In place of (2.27), we get

$$S_3(m, \delta) \leq \frac{2+m\delta}{2} \left(\left\{ \frac{1}{2\pi} - q(T_2) \right\} \int_A^{T_2} \phi_0(y) \log \frac{y}{2\pi} dy + E_1 \right) \quad (2.52)$$

where

$$\begin{aligned} E_1 &= \{N(T_2) - F(T_2) + R(T_2)\} + \phi_0(T_2) - \{N(A) - F(A) + R(A)\} \phi_0(A) \\ &< 2R(T_2) \phi_0(T_2) \end{aligned} \quad (2.53)$$

and $R(T) = 0.137 \log T + 0,443 \log \log T + 1.588$. Putting

$$V'' = \frac{X^2}{\log T_2}$$

we have

$$V'' = \frac{X^2}{\nu X} = \frac{X}{\nu} = X \left\{ \frac{(1-\nu)^2}{\nu} + 2 - \nu \right\} = Y + 2X - \nu X, \quad (2.54)$$

where

$$Y = X \frac{(1-\nu)^2}{\nu} \quad (2.55)$$

Proceeding as in (2.28) and (2.31) and using (2.53), we find

$$\begin{aligned} S_3(m, \delta) &< \frac{2+m\delta}{2} \frac{1}{2\pi} e^{-V''} \{X^4(V'')^{-3} - (\log 2\pi) X^2(V'')^{-2}\} + \frac{2+m\delta}{2} E_1 \\ &< \frac{2+m\delta}{4\pi} e^{-Y-2X} T_2 X G_0 + (2+m\delta) R(T_2) \phi_0(T_2) \end{aligned} \quad (2.56)$$

where

$$G_0 = \nu^2 \left(\nu - \frac{\log 2\pi}{X} \right) \quad (2.57)$$

As $R(y)/\log y$ decrease for $y > e^e$, we have

$$\begin{aligned} R(T_2) \phi_0(T_2) &= \frac{R(T_2)}{\log T_2} \phi_0(T_2) \log T_2 \leq \frac{R(A)}{\log A} \phi_0(T_2) \log T_2 \\ &\leq \frac{R(A)}{\log A} \frac{e^{-V''}}{T_2} \log T_2 = \frac{R(A)}{\log A} \frac{e^{-Y-2X+\nu X}}{T_2} \log T_2 \\ &= \frac{R(A)}{\log A} e^{-Y-2X} \log T_2. \end{aligned}$$

by (2.54). Then (2.50) and (2.51) yield

$$\log T_2 = \nu X \leq X$$

hence

$$R(T_2) \phi_0(T_2) < 0.24471 X e^{-Y-2X}. \quad (2.58)$$

We have

$$S_4(m, \delta) \leq \frac{R_m(\delta)}{\delta^m} \left(\left\{ \frac{1}{2\pi} + q(T_2) \right\} \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy + E_0 \right) \quad (2.59)$$

where

$$\begin{aligned} E_0 &= \{R(T_2) + F(T_2) - N(T_2)\} \phi_m(T_2) \\ &< 2R(T_2) \phi_m(T_2) = 2R(T_2) \phi_0(T_2) T^{-m} \end{aligned} \quad (2.60)$$

Also

$$\int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy = \frac{z^2}{2m^2} \{K_2(z, U') - \frac{2m \log 2\pi}{z} K_1(z, U')\} \quad (2.61)$$

where $z = 2X\sqrt{m}$ and

$$U' = \frac{2m}{z} \log T_2 = \frac{2m}{2X\sqrt{m}} \log T_2 = \sqrt{m} \frac{\log T_2}{X} = \sqrt{m} \frac{\nu X}{X} = \nu \sqrt{m}$$

By assuming

$$\nu > \frac{1}{\sqrt{m}} \quad (2.62)$$

we have $U' > 1$; also $m \geq 2$ since $\nu \leq 1$.

By Lemma 2.32 and Corollary 2.34

$$\begin{aligned} K_2(z, U') - \frac{2m \log 2\pi}{z} K_1(z, U') &< K_2(z, U') < \left(U' + \frac{2}{z}\right) Q_1(z, U') \\ &= \sqrt{m} \left(\nu + \frac{1}{mX}\right) \frac{U'^2}{z(U'^2 - 1)} e^{-\frac{1}{2}z(U' + 1/U')} \end{aligned}$$

Now

$$\frac{1}{2}z \left(U' + \frac{1}{U'}\right) = X\sqrt{m} \left(\nu\sqrt{m} + \frac{1}{\nu\sqrt{m}}\right) = m\nu X + (Y + 2X - \nu X)$$

Hence,

$$\begin{aligned} K_2(z, U') &< \sqrt{m} \left(\nu + \frac{1}{mX}\right) \frac{m\nu^2}{2X\sqrt{m}(m\nu^2 - 1)} e^{-m\nu X - (Y + 2X - \nu X)} \\ &= \left(\nu + \frac{1}{mX}\right) \frac{m}{2(m-1)} \frac{(m-1)\nu^2}{(m\nu^2 - 1)} X^{-1} T_2^{-(m-1)} e^{-Y - 2X} \\ &= G_1 \frac{m}{2(m-1)} X^{-1} T_2^{-(m-1)} e^{-Y - 2X} \end{aligned} \quad (2.63)$$

where

$$G_1 = \frac{(m-1)\nu^2}{(m\nu^2 - 1)} \left(\nu + \frac{1}{mX}\right) \quad (2.64)$$

Then

$$\begin{aligned} \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy &< \frac{z^2}{2m^2} G_1 \frac{m}{2(m-1)} X^{-1} T_2^{-(m-1)} e^{-Y - 2X} \\ &= \frac{1}{m-1} G_1 X T_2^{-(m-1)} e^{-Y - 2X} \end{aligned}$$

We define

$$G_2 = \frac{R_m(\delta)}{2^m} \{1 + 2\pi q(T_2)\} = \{1 + 2\pi q(T_2)\} \left\{ \frac{(1 + \delta)^{m+1} + 1}{2} \right\}^m \quad (2.65)$$

Then

$$\begin{aligned}
S_4(m, \delta) &\leq \frac{R_m(\delta)}{2^m} \left(\frac{2}{\delta}\right)^m \frac{1}{2\pi} \{1 + 2\pi q(T_2)\} \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{2^m} \left(\frac{2}{\delta}\right)^m E_0 \\
&= \left(\frac{2}{\delta}\right)^m \frac{1}{2\pi} G_2 \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{2^m} \left(\frac{2}{\delta}\right)^m E_0 \\
&< \left(\frac{2}{\delta}\right)^m \frac{1}{2\pi} G_2 \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy + G_2 \left(\frac{2}{\delta}\right)^m E_0 \\
&< \left(\frac{2}{\delta}\right)^m \frac{1}{2\pi(m-1)} G_2 G_1 X T_2^{-(m-1)} e^{-Y-2X} + G_2 \left(\frac{2}{\delta}\right)^m E_0
\end{aligned}$$

Now $1 + m\delta < R_m(\delta)/2^m < G_2$. We obtain

$$\begin{aligned}
S_3(m, \delta) + S_4(m, \delta) &< \frac{1}{2\pi} G_2 X e^{-Y-2X} \left\{ G_0 T_2 + \frac{1}{m-1} G_1 \left(\frac{2}{\delta T_2}\right)^m T_2 \right\} \\
&\quad + 2G_2 R(T_2) \phi_0(T_2) \left\{ 1 + \left(\frac{2}{\delta T_2}\right)^m \right\}
\end{aligned}$$

If G_0 and G_1 were independent of ν , and hence of T_2 , then the expression inside the first braces would be minimized by choosing

$$T_2 = \frac{2}{\delta} \left(\frac{G_1}{G_0}\right)^{1/m} \quad (2.66)$$

Postponing the reconciliation of this with the previous definition of T_2 , we obtain

$$\begin{aligned}
S_3(m, \delta) + S_4(m, \delta) + \frac{1}{2}m\delta &< \frac{1}{2\pi} G_2 X e^{-Y-2X} \left\{ G_2 T_2 + \frac{1}{m-1} G_1 \left(\frac{2}{\delta}\right)^m T_2^{1-m} \right\} \\
&\quad + 2G_2 R(T_2) \phi_0(T_2) \left\{ 1 + \left(\frac{2}{\delta T_2}\right)^m \right\} + \frac{1}{2}mG_2\delta \\
&= \frac{1}{2\pi} G_2 X e^{-Y-2X} \left\{ \frac{m}{m-1} G_0^{1-1/m} G_1^{1/m} \frac{2}{\delta} \right\} \\
&\quad + 2G_2 R(T_2) \phi_0(T_2) \left(1 + \frac{G_0}{G_1} \right) + \frac{1}{2}mG_2\delta \\
&= \frac{1}{2}mG_2 \left\{ \frac{2}{\pi(m-1)} G_0^{1-1/m} G_1^{1/m} X e^{-Y-2X} \frac{1}{\delta} + \delta \right\} \\
&\quad + 2G_2 R(T_2) \phi_0(T_2) \left(1 + \frac{G_0}{G_1} \right)
\end{aligned}$$

The expression inside the last braces is minimized by choosing

$$\delta = \left\{ \frac{2}{\pi(m-1)} G_0^{1-1/m} G_1^{1/m} e^{-Y} \right\}^{1/2} X^{1/2} e^{-X} \quad (2.67)$$

so that (2.66) becomes

$$T_2 = \frac{2}{\delta} \left(\frac{G_1}{G_0}\right)^{1/m} = \left(\frac{G_1}{G_0}\right)^{1/2m} \left\{ \frac{2\pi(m-1)}{G_0} e^Y \right\}^{1/2} X^{-1/2} e^X \quad (2.68)$$

Moreover, since $R(T_2)\phi_0(T_2) < 0.24471Xe^{-Y-2X}$,

$$S_3(m, \delta) + S_4(m, \delta) + \frac{1}{2}m\delta < G_2 \left\{ \frac{2}{\pi} G_0^{1-1/m} G_1^{1/m} e^{-Y} \right\}^{1/2} \frac{m}{\sqrt{m-1}} X^{1/2} e^{-X} \\ + 0.48942 G_2 \left(1 + \frac{G_0}{G_1} \right) e^{-Y} X e^{-2X}$$

The coefficient $\frac{m}{\sqrt{m-1}}$ in the next to the last term is minimized by choosing $m = 2$. For this value we obtain

$$\delta = (G_0 G_1)^{1/4} \sqrt{\frac{2}{\pi}} e^{-Y/2} X^{1/2} e^{-X} \quad (2.69)$$

$$T_2 = \left(\frac{G_1}{G_0^3} \right)^{1/4} \sqrt{2\pi} e^{Y/2} X^{-1/2} e^X \quad (2.70)$$

$$G_1 = \frac{\nu^2}{2\nu^2 - 1} \left(\nu + \frac{1}{2X} \right) \quad (2.71)$$

Also

$$S_3(2, \delta) + S_4(2, \delta) + \delta < G_2 (G_0 G_1)^{1/4} e^{-Y/2} \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} \\ + 0.48942 G_2 \left(1 + \frac{G_0}{G_1} \right) e^{-Y} X e^{-2X} \quad (2.72)$$

provided the choice of T_2 in (2.70) is consistent with (2.50) and provided both (2.51) and (2.62) hold when $m = 2$.

We readily see that T_2 of $T_2 = e^{\nu X}$ and T_2 just above are equal if and only if ν is such that $k(\nu) = 1$ where

$$k(\nu) = \frac{1}{2\pi} X \left(\frac{G_0^3}{G_1} \right)^{1/2} e^{-2X(1-\nu)} e^{-X(1-\nu)^2/\nu} \quad (2.73)$$

If $1/\sqrt{2} < \nu \leq \sqrt{3/2}$, it is not hard to see that G_1 decreases as ν increases, G_0 is also increasing function of ν . Hence, $k(\nu)$ is strictly increasing for increasing $\nu \in (1/\sqrt{2}, 1]$. Now $k(\nu) \rightarrow 0$ as $\nu \rightarrow 1/\sqrt{2}$ from the right; and we easily see that $k(1) > 1$ (for all $X \geq 10$). As a result there is a unique $\nu \in (1/\sqrt{2}, 1)$ such that $k(\nu) = 1$. Henceforth, let ν be this number so that ν depends on X ; then G_0 , G_1 , Y and T_2 are defined in terms of ν by (7.12), (7.25), (7.10) and (7.5), (7.24b). Of course, (7.17) holds since $m = 2$. Hence (7.26) will be fully established once it is shown that $T \geq A$. We have, for $1/\sqrt{2} < \nu \leq 1$,

$$H(\nu) \equiv \frac{G_0^3}{G_1} = \nu^4 (2\nu^2 - 1) \frac{(\nu - \log(2\pi)/X)^3}{\nu + 1/(2X)} \begin{cases} < (\nu - \log(2\pi)/X)^2, & \text{all } X; \\ > 0.37\nu^6(2\nu^2 - 1), & X \geq 10. \end{cases} \quad (2.74)$$

If we define for $j = 0$ and 1,

$$\nu_j = 1 - \frac{1}{2X} \log \frac{X}{(2 + 3j)\pi} \quad (2.75)$$

we see that

$$H(\nu_0) < 1, \quad (X \geq \frac{1}{2\pi}), \quad H(\nu_1) > 0.22318, \quad (X > 0)$$

Inasmuch as

$$\begin{aligned} k(\nu_j) &= \frac{1}{2\pi} X H(\nu_j)^{1/2} e^{-2X(1-\nu)} e^{-X(1-\nu)^2/\nu} \\ &= \frac{1}{2\pi} X H(\nu_j)^{1/2} \exp \left\{ -\log \frac{X}{(2+3j)\pi} \right\} \exp \left\{ -\frac{1}{4\nu_j X} \log^2 \frac{X}{(2+3j)\pi} \right\} \\ &= \frac{2+3j}{2} H(\nu_j)^{1/2} \exp \left\{ -\frac{1}{4\nu_j X} \log^2 \frac{X}{(2+3j)\pi} \right\} \end{aligned}$$

we see that

$$k(\nu_0) < 1 = k(\nu), \quad (X \geq \frac{1}{2\pi}), \quad k(\nu_1) > 1 = k(\nu), \quad (X > 0)$$

So

$$\nu_0 < \nu, \quad (\log x \geq 0.145), \quad \nu < \nu_1, \quad (\log x > 0) \quad (2.76)$$

Of course $\nu_0 < \nu_1$ in all cases. For $\log x \geq 4890$, we get

$$T_2 > e^{\nu_0 X} > A$$

Hence,

$$\begin{aligned} S_3(2, \delta) + S_4(2, \delta) + \delta &< G_2(G_0 G_1)^{1/4} e^{-Y/2} \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} \\ &+ 0.48942 G_2 \left(1 + \frac{G_0}{G_1} \right) e^{-Y} X e^{-2X} \end{aligned}$$

for $\log x \geq 4890$; and for these x , we have $\nu_0 > 0.9737 > \sqrt{15/16}$. It is a simple matter to verify that

$$\frac{G_0}{G_1} < 2\nu^2 - 1 < 1, \quad Y < X(1 - \nu_0)^2/\nu_0 < 0.025 \quad (2.77)$$

$$G_0 G_1 = \frac{\nu^6}{2\nu^2 - 1} \left(1 - \frac{\log 2\pi}{\nu X} \right) \left(1 + \frac{1}{2\nu X} \right) \begin{cases} < (1 - \frac{3}{2\nu X})(1 + \frac{1}{2\nu X}) < 1, & X \geq 11; \\ > \frac{1}{2}\nu^6/(2\nu^2 - 1) \geq 0.84375, & X \geq 11. \end{cases} \quad (2.78)$$

Then

$$\begin{aligned} S_3(2, \delta) + S_4(2, \delta) + \delta &< G_2(G_0 G_1)^{1/4} e^{-Y/2} \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} \\ &+ 0.48942 G_2 \left(1 + \frac{G_0}{G_1} \right) e^{-Y} X e^{-2X} \\ &< G_2(G_0 G_1)^{1/4} e^{-Y/2} \left\{ \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} + 1.022 X e^{-2X} \right\} \end{aligned} \quad (2.79)$$

Taking $T_1 = 0$ and using Proposition 2.3, we obtain

$$\begin{aligned}
\frac{1}{\sqrt{x}}\{S_1(2, \delta) + S_2(2, \delta)\} &< \frac{1}{\sqrt{x}} \frac{R_2(\delta)}{\delta^2} \sum_{\gamma} \frac{1}{\gamma^3} < \frac{1}{\sqrt{x}} G_2 \left(\frac{2}{\delta} \right)^2 (0.00146435) \\
&< \frac{4}{\sqrt{x}} G_2(0.00146435) (G_0 G_1)^{-1/2} \frac{\pi}{2} e^Y X^{-1} e^{2X} \\
&< 0.009434 \frac{1}{\sqrt{x}} G_2 (G_0 G_1)^{-1/2} X^{-1} e^{2X} \\
&< 0.010715 \frac{1}{\sqrt{x}} G_2 (G_0 G_1)^{1/4} X^{-1} e^{2X}
\end{aligned}$$

Putting

$$\Omega = \frac{1}{\sqrt{x}}\{S_1(2, \delta) + S_2(2, \delta)\} + S_3(2, \delta) + S_4(2, \delta) + \delta$$

we obtain that for $\log x \geq 4890$

$$\Omega < G_2(G_0 G_1)^{1/4} e^{-Y/2} \left\{ \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} + 1.022 X e^{-2X} + 0.010715 \frac{1}{\sqrt{x}} X^{-1} e^{2X} \right\} \quad (2.80)$$

Since

$$\frac{1}{x} |\psi(x) - x| < \frac{1}{x} \left\{ \log 2\pi + \frac{1}{2} \log(1 - 1/x^2) \right\} + \Omega < \Omega + \frac{\log 2\pi}{x}$$

Now

$$\begin{aligned}
\frac{1}{x} |\vartheta(x) - x| &< \Omega + \frac{\log 2\pi}{x} + \frac{1.43}{\sqrt{x}} \\
&< \Omega + \frac{0.000085}{\sqrt{x}} G_2(G_0 G_1)^{1/4} e^{-Y/2} X^{-1} e^{2X}
\end{aligned}$$

Hence,

$$\frac{1}{x} |\psi(x) - x|, \quad \frac{1}{x} |\vartheta(x) - x| < G_3 \sqrt{\frac{8}{\pi}} X^{1/2} e^{-X} = G_3 \varepsilon_0(x) \quad (2.81)$$

for $\log x \geq 4890$, where

$$\begin{aligned}
G_3 &= G_2(G_0 G_1)^{1/4} e^{-Y/2} \left\{ 1 + \sqrt{\frac{\pi}{8}} (1.022 X^{1/2} e^{-X} + \frac{0.0108}{\sqrt{x}} X^{-3/2} e^{3X}) \right\} \\
&< G_2(G_0 G_1)^{1/4} e^{-Y/2} \left\{ 1 + 0.65 X^{1/2} e^{-X} \right\} \\
&< G_2(G_0 G_1)^{1/4} e^{-Y/2} \exp \left\{ 0.65 X^{1/2} e^{-X} \right\}
\end{aligned} \quad (2.82)$$

Also by definition of $q(y)$, relation for T_2 , and $H(\nu)$

$$\begin{aligned}
1 + 2\pi q(T_2) &= 1 + 2\pi \frac{0.137 + 0.443/\log T_2}{T_2 \log(T_2/2\pi)} \\
&< 1 + \frac{2\pi}{T_2} \frac{0.137 + 0.443/\log A}{\log(A/2\pi)} \\
&< 1 + 0.0057154 \frac{2\pi}{T_2} \\
&= 1 + 0.0057154 \sqrt{2\pi} \left(\frac{G_0^3}{G_1} \right)^{1/4} e^{-Y/2} X^{1/2} e^{-X} \\
&< 1 + 0.0143265 X^{1/2} e^{-X} \\
&< \exp(0.0143265 X^{1/2} e^{-X})
\end{aligned}$$

Further,

$$\begin{aligned}\frac{R_2(\delta)}{2^2} &= \left\{ \frac{(1+\delta)^3 + 1}{2} \right\}^2 = \left\{ 1 + \frac{1}{2}\delta(3 + 3\delta + \delta^2) \right\}^2 < \left(1 + \frac{3.01}{2}\delta \right)^2 \\ &< \left\{ \exp\left(\frac{3.01}{2}\delta\right) \right\}^2 = \exp(3.01\delta) < \exp(2.402X^{1/2}e^{-X})\end{aligned}$$

Then

$$\begin{aligned}G_3 &< G_2(G_0G_1)^{1/4}e^{-Y/2}\exp\left\{0.65X^{1/2}e^{-X}\right\} \\ &< \exp(2.402X^{1/2}e^{-X})\exp(0.0143265X^{1/2}e^{-X})(G_0G_1)^{1/4}e^{-Y/2}\exp\left\{0.65X^{1/2}e^{-X}\right\} \\ &< (G_0G_1)^{1/4}e^{-Y/2}\exp(3.67X^{1/2}e^{-X}) = \left\{ G_0G_1e^{-2Y}\exp(14.68X^{1/2}e^{-X}) \right\}^{1/4}\end{aligned}$$

By (2.57), we obtain for $\log x \geq 4890$

$$\begin{aligned}\frac{X}{\nu^2}G_0\exp(14.68X^{1/2}e^{-X}) &= (\nu X - \log 2\pi)\exp(14.68X^{1/2}e^{-X}) \\ &< (\nu X - \log 2\pi)(1 + 14.69X^{1/2}e^{-X}) \\ &= \nu X - \log 2\pi + (\nu X - \log 2\pi)14.69X^{1/2}e^{-X} \\ &< \nu X - \log 2\pi + 5 \cdot 10^{-10} < \nu X - 1 = X(\nu - 1/X)\end{aligned}$$

Hence, (2.71) yields

$$\begin{aligned}G_3 &< \left\{ G_0G_1e^{-2Y}\exp(14.68X^{1/2}e^{-X}) \right\}^{1/4} < \left\{ \nu^2\left(\nu - \frac{1}{X}\right)G_1e^{-2Y} \right\}^{1/4} \\ &= \left\{ \frac{\nu^4}{2\nu^2 - 1}\left(\nu - \frac{1}{X}\right)\left(\nu + \frac{1}{2X}\right)e^{-2Y} \right\}^{1/4}\end{aligned}$$

As a result of (2.81) one deduce for $\log x \geq 4890$

$$|\psi(x) - x|, \quad |\vartheta(x) - x| < x\varepsilon_0(x)M(\nu)L(\nu) \quad (2.83)$$

where

$$L(\nu) = \left\{ \frac{\nu^6}{2\nu^2 - 1} \right\}^{1/4} \quad (2.84)$$

$$M(\nu) = \left\{ \left(1 - \frac{1}{\nu X}\right)\left(1 + \frac{1}{2\nu X}\right)e^{-2X(1-\nu)^2/\nu} \right\}^{1/4} \quad (2.85)$$

The function $L(\nu)$ is real valued for $\nu > 1/\sqrt{2}$ and, as is easily seen, has a minimum value at $\nu = \sqrt{3/4}$. If $\log x \geq 0.145$, then $\nu > \nu_0 > 1/\sqrt{2}$. Hence,

$$\left(\frac{27}{32}\right)^{1/4} < L(\nu) < 1, \quad (x \geq e^{0.145}) \quad (2.86)$$

In addition,

$$\begin{aligned}M(\nu) &< \exp \frac{1}{4} \left\{ -\frac{1}{\nu X} + \frac{1}{2\nu X} - \frac{2X}{\nu}(1-\nu)^2 \right\} \\ &< \exp \frac{1}{4} \left\{ -\frac{1}{\nu X} + \frac{1}{2\nu X} - \frac{2X}{\nu}(1-\nu_1)^2 \right\} \\ &= \exp \frac{1}{4} \left\{ -\frac{1}{2\nu X} \left(1 + \log^2 \frac{X}{5\pi} \right) \right\} < E(x)\end{aligned} \quad (2.87)$$

where

$$E(x) = \exp \frac{1}{4} \left\{ -\frac{1}{2\nu_1 X} \left(1 + \log^2 \frac{X}{5\pi} \right) \right\} = \exp \frac{1}{4\nu_1} \left\{ -\frac{1}{2X} - 2X(1 - \nu_1)^2 \right\} \quad (2.88)$$

It is clear from the first part of (2.88) that $E(x) < 1$ for all x . So for $\log x \geq 4890$

$$|\psi(x) - x|, \quad |\vartheta(x) - x| < x\varepsilon_0(x).$$

For smaller x we use the table and relation $0 \leq \psi(x) - \vartheta(x) < 1.427\sqrt{x}$. □

2.6.1 Numerical Bounds for Moderate Values of x

Let

$$T_0 = \frac{1}{\delta} \left(\frac{2R_m(\delta)}{2 + m\delta} \right)^{1/m}$$

and leave T_1 unspecified for the moment. We showed that by letting $T_2 = e^{\nu X}$,

$$S_3(m, \delta) < \frac{2 + m\delta}{2} \left(\left\{ \frac{1}{2\pi} - q(T_2) \right\} \int_A^{T_2} \phi_0(y) \log \frac{y}{2\pi} dy + 2R(T_2)\phi_0(T_2) \right)$$

$$S_4(m, \delta) < \frac{R_m(\delta)}{\delta^m} \left(\left\{ \frac{1}{2\pi} + q(T_2) \right\} \int_{T_2}^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy + 2R(T_2)\phi_0(T_2)T^{-m} \right)$$

so that

$$S_3(m, \delta) + S_4(m, \delta) < \frac{1}{2\pi} h_3(T_2) + e_3(T_2)$$

where

$$h_3(T) = \frac{2 + m\delta}{2} \int_A^T \phi_0(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{\delta^m} \int_T^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy$$

and

$$\begin{aligned} e_3(T) = & q(T) \left\{ -\frac{2 + m\delta}{2} \int_A^T \phi_0(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{\delta^m} \int_T^{\infty} \phi_m(y) \log \frac{y}{2\pi} dy \right\} \\ & + R(T)\phi_0(T) \{ 2 + m\delta + 2 \frac{R_m(\delta)}{(\delta T)^m} \} \end{aligned}$$

The situation for $S_1(m, \delta) + S_2(m, \delta)$ is entirely similar. If we leave T_1 and D unspecified but subject to $2 \leq D \leq A$ and $T_1 \geq D$, then we get

$$S_1(m, \delta) + S_2(m, \delta) < \frac{1}{\pi} h_1(T_1) + e_1(T_1)$$

where

$$\begin{aligned} h_1(T) = & \frac{2 + m\delta}{2} \int_D^T \frac{1}{y} \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{\delta^m} \int_T^{\infty} \frac{1}{y^{m+1}} \log \frac{y}{2\pi} dy \\ & + (2 + m\delta)\pi \left\{ G(D) + \frac{1}{4\pi} \log^2 \frac{D}{2\pi} \right\} \end{aligned}$$

$$\begin{aligned} G(D) = & \sum_{0 < \gamma \leq D} \frac{1}{(\gamma^2 + 1/4)^{1/2}} - \frac{1}{4\pi} \left\{ \left(\log \frac{D}{2\pi} - 1 \right)^2 + 1 \right\} \\ & + \frac{1}{D} \left\{ 0.137 \log D + 0.443 \left(\log \log D + \frac{1}{\log D} \right) + 2.6 - N(D) \right\} \end{aligned}$$

$$e_1(T) = -\frac{2\pi}{T} \left\{ \frac{2+m\delta}{2} \left(0.137 + \frac{0.443}{\log D} \right) - \frac{R_m(\delta)}{(m+1)(\delta T)^m} \left(0.137 + \frac{0.443}{\log T} \right) \right\} \\ + \frac{2\pi}{T} \left\{ \frac{2+m\delta}{2} - \frac{R_m(\delta)}{(\delta T)^m} \right\} \{N(T) - F(T) - R(T)\}$$

Let

$$C(D) = 4\pi \left(0.137 + \frac{0.443}{\log D} \right), \quad S(D) = \sum_{0 < \gamma \leq D} \frac{1}{(\gamma^2 + 1/4)^{1/2}}$$

Theorem 2.36. *Let T_0 be defined as above and satisfy $T_0 \geq D$, where $2 \leq A$. Let m be a positive integer and let $\delta > 0$. Then*

$$S_1(m, \delta) + S_2(m, \delta) < \Omega_1^*$$

where

$$\Omega_1^* = \frac{2+m\delta}{4\pi} \left\{ \left(\log \frac{T_0}{2\pi} + \frac{1}{m} \right)^2 + 4\pi G(D) + \frac{1}{m^2} - \frac{mC(D)}{(m+1)T_0} \right\}$$

where $G(D)$ and $C(D)$ are defined above. Moreover, if

$$\Omega_3^* = \frac{1}{2\pi} h_3(T_2) + e_3(T_2)$$

then

$$|\psi(x) - x| < \varepsilon_0^* x, \quad (x \geq e^b)$$

where

$$\varepsilon_0^* = \Omega_1^* e^{-b/2} + \Omega_3^* + \frac{m}{2} \delta + e^{-b} \log 2\pi$$

Remark 2.37. After doing this article we realized that a similar theorem to Theorem 2.35 for $|\psi(x) - x|$ was done by P. Dusart. But as you may see in this article we give the proofs with more details.

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in this case for $b \geq 5765$

$$\varepsilon < \Omega_1 e^{-b/2} + \Omega_2 + md/2 + \log(2\pi)e^{-b}$$

For $b = \log(8 \cdot 10^{11}) \approx 27.4079$, we have $m = 1$, $\delta = 9(-6)$ and $\varepsilon = 2.84888(-5)$. For $b = \log 10^{16} \approx 36.8414$, we have $m = 2$, $\delta = 5.24(-8)$ and $\varepsilon = 4.66629(-7)$.

Table 3: $|\psi(x) - x| < x\varepsilon$, $(x \geq e^b)$, $\varepsilon = CX^{3/4}e^{-X}$

b	m	δ	ε	b	m	δ	ε
18.42	1	4.77(-4)	1.14853(-3)	950	21	2.15(-12)	2.36304(-11)
18.43	1	4.75(-4)	1.14399(-3)	1000	21	2.12(-12)	2.32993(-11)
18.44	1	4.73(-4)	1.13947(-3)	1050	21	2.09(-12)	2.29730(-11)
18.45	1	4.71(-4)	1.13496(-3)	1100	20	2.16(-12)	2.26446(-11)
18.5	1	4.61(-4)	1.11269(-3)	1150	20	2.13(-12)	2.23136(-11)
18.7	1	4.22(-4)	1.02778(-3)	1200	20	2.09(-12)	2.19866(-11)
19.0	1	3.69(-4)	9.12089(-4)	1250	19	2.17(-12)	2.16638(-11)
19.5	1	2.96(-4)	7.46822(-4)	1300	19	2.13(-12)	2.13312(-11)
20	1	2.37(-4)	6.10849(-4)	1350	19	2.10(-12)	2.10036(-11)
21	1	1.52(-4)	4.07427(-4)	1400	19	2.07(-12)	2.06817(-11)
22	1	9.68(-5)	2.70724(-4)	1450	18	2.14(-12)	2.03545(-11)
23	1	6.17(-5)	1.79271(-4)	1500	18	2.11(-12)	2.00263(-11)
24	1	3.93(-5)	1.18353(-4)	1550	18	2.07(-12)	1.97041(-11)
25	1	2.51(-5)	7.7946(-5)	1600	17	2.15(-12)	1.93833(-11)
26	1	1.61(-5)	5.12658(-5)	1650	17	2.12(-12)	1.90539(-11)
27	1	1.06(-5)	3.37472(-5)	1700	17	2.08(-12)	1.87300(-11)
28	2	3.18(-6)	2.22937(-5)	1750	17	2.05(-12)	1.84125(-11)
29	2	2.00(-6)	1.45047(-5)	1800	16	2.13(-12)	1.80865(-11)
30	2	1.26(-6)	9.41621(-6)	1850	16	2.09(-12)	1.77615(-11)
35	2	1.22(-7)	1.05487(-6)	1900	16	2.05(-12)	1.74427(-11)
40	3	7.81(-9)	1.16299(-7)	1950	15	2.14(-12)	1.71251(-11)
45	4	5.59(-10)	1.23376(-8)	2000	15	2.10(-12)	1.67987(-11)
50	7	3.44(-11)	1.30131(-9)	2100	15	2.02(-12)	1.61646(-11)
75	26	2.20(-12)	2.96691(-11)	2200	14	2.07(-12)	1.55206(-11)
100	26	2.18(-12)	2.94551(-11)	2300	13	2.13(-12)	1.48933(-11)
150	26	2.15(-12)	2.90681(-11)	2400	13	2.04(-12)	1.42535(-11)
200	26	2.13(-12)	2.87042(-11)	2500	12	2.10(-12)	1.36270(-11)
250	25	2.18(-12)	2.83496(-11)	2600	12	2.00(-12)	1.29976(-11)
300	25	2.15(-12)	2.79972(-11)	2700	11	2.06(-12)	1.23732(-11)
350	25	2.13(-12)	2.76518(-11)	3000	10	1.91(-12)	1.05302(-11)
400	25	2.10(-12)	2.73130(-11)	3200	9	1.86(-12)	9.32308(-12)
450	24	2.16(-12)	2.69688(-11)	3500	7	1.88(-12)	7.53751(-12)
500	24	2.13(-12)	2.66291(-11)	3700	6	1.83(-12)	6.39612(-12)
550	24	2.10(-12)	2.62963(-11)	4000	5	1.60(-12)	4.78674(-12)
600	23	2.16(-12)	2.59588(-11)	4200	4	1.51(-12)	3.77702(-12)
650	23	2.14(-12)	2.56227(-11)	4500	3	1.23(-12)	2.46504(-12)
700	23	2.11(-12)	2.52897(-11)	4700	2	1.18(-12)	1.77185(-12)
750	22	2.17(-12)	2.49584(-11)	5000	2	6.51(-13)	9.76476(-13)
800	22	2.14(-12)	2.46224(-11)	5200	2	4.38(-13)	6.56727(-13)
850	22	2.11(-12)	2.42914(-11)	5500	2	2.42(-13)	3.62532(-13)
900	22	2.08(-12)	2.39657(-11)	5700	2	1.63(-13)	2.44112(-13)

Table 4: η for the case $\varepsilon = CX^{3/4}e^{-X}$

b	η_1	η_2	η_3	η_4
18.42	0.0211674	0.390116	7.18984	132.509
18.43	0.0210952	0.388996	7.17308	132.272
18.44	0.0210232	0.387878	7.15635	132.035
18.45	0.0209968	0.388441	7.18616	132.944
18.5	0.0208073	0.389097	7.27612	136.063
18.7	0.0195278	0.371028	7.04952	133.941
19	0.0177857	0.346822	6.76302	131.879
19.5	0.0149364	0.298729	5.97458	119.492
20	0.0128278	0.269384	5.65707	118.799
21	0.0089634	0.197195	4.33829	95.4423
22	0.00622665	0.143213	3.2939	75.7596
23	0.00430251	0.10326	2.47825	59.478
24	0.00295883	0.0739707	1.84927	46.2317
25	0.0020266	0.0526915	1.36998	35.6195
26	0.00138418	0.0373728	1.00907	27.2448
27	0.000944921	0.0264578	0.740818	20.7429
28	0.000646516	0.018749	0.54372	15.7679
29	0.00043514	0.0130542	0.391626	11.7488
30	0.000329567	0.0115349	0.40372	14.1302
35	0.0000421946	0.00168778	0.0675114	2.70046
40	5.81494(-6)	0.000290747	0.0145374	0.726868
50	9.7598(-8)	7.31985(-6)	0.000548989	0.0411742
75	2.96691(-9)	2.96691(-7)	0.0000296691	0.00296691
100	4.41826(-9)	6.6274(-7)	0.000099411	0.0149116
150	5.81362(-9)	1.16272(-6)	0.000232545	0.046509
200	7.17604(-9)	1.79401(-6)	0.000448502	0.112126
250	8.50488(-9)	2.55146(-6)	0.000765439	0.229632
300	9.79902(-9)	3.42966(-6)	0.00120038	0.420133
350	1.10607(-8)	4.42429(-6)	0.00176972	0.707886
400	1.22909(-8)	5.53089(-6)	0.0024889	1.12001
450	1.34844(-8)	6.74219(-6)	0.00337109	1.68555
500	1.4646(-8)	8.0553(-6)	0.00443041	2.43673
550	1.57778(-8)	9.46665(-6)	0.00567999	3.408
600	1.68732(-8)	0.0000109676	0.00712893	4.6338
650	1.79359(-8)	0.0000125551	0.00878859	6.15201
700	1.89673(-8)	0.0000142255	0.0106691	8.00183
750	1.99667(-8)	0.0000159734	0.0127787	10.223
800	2.0929(-8)	0.0000177897	0.0151212	12.853
850	2.18623(-8)	0.0000196761	0.0177085	15.9376
900	2.27674(-8)	0.000021629	0.0205476	19.5202

For $b \geq 5213$,

$$\varepsilon < \Omega_1^* e^{-b/2} + \Omega_2^* + m\delta/2 + \log(2\pi)e^{-b}$$

so that instead of Ω^* we use general theorem $\varepsilon = \sqrt{8/\pi} \dots$). $D=2500$, minimize all:

Table 5: η for the case $\varepsilon = CX^{3/4}e^{-X}$

b	η_1	η_2	η_3	η_4
950	2.36304(-8)	0.0000236304	0.0236304	23.6304
1000	2.44643(-8)	0.0000256875	0.0269719	28.3204
1050	2.52703(-8)	0.0000277973	0.030577	33.6347
1100	2.60413(-8)	0.0000299475	0.0344396	39.6055
1150	2.67764(-8)	0.0000321317	0.038558	46.2696
1200	2.74833(-8)	0.0000343541	0.0429427	53.6783
1250	2.81629(-8)	0.0000366118	0.0475953	61.8739
1300	2.87972(-8)	0.0000388762	0.0524828	70.8518
1350	2.94051(-8)	0.0000411671	0.057634	80.6876
1400	2.99885(-8)	0.0000434834	0.0630509	91.4238
1450	3.05317(-8)	0.0000457975	0.0686963	103.044
1500	3.10407(-8)	0.0000481131	0.0745754	115.592
1550	3.15265(-8)	0.0000504425	0.080708	129.133
1600	3.19825(-8)	0.0000527712	0.0870724	143.669
1650	3.23916(-8)	0.0000550657	0.0936117	159.14
1700	3.27774(-8)	0.0000573605	0.100381	175.667
1750	3.31425(-8)	0.0000596565	0.107382	193.287
1800	3.346(-8)	0.0000619011	0.114517	211.856
1850	3.37469(-8)	0.000064119	0.121826	231.47
1900	3.40132(-8)	0.0000663258	0.129335	252.204
1950	3.42502(-8)	0.0000685004	0.137001	274.002
2000	3.52772(-8)	0.0000740822	0.155573	326.703
2100	3.55621(-8)	0.0000782365	0.17212	378.665
2200	3.56974(-8)	0.0000821041	0.188839	434.331
2300	3.57438(-8)	0.0000857851	0.205884	494.122
2400	3.56337(-8)	0.0000890842	0.22271	556.776
2500	3.54302(-8)	0.0000921186	0.239508	622.722
2600	3.50936(-8)	0.0000947527	0.255832	690.747
2700	3.71195(-8)	0.000111358	0.334075	1002.23
3000	3.36966(-8)	0.000107829	0.345053	1104.17
3200	3.26308(-8)	0.000114208	0.399727	1399.04
3500	2.78888(-8)	0.000103189	0.381798	1412.65
3700	2.55845(-8)	0.000102338	0.409352	1637.41
4000	2.01043(-8)	0.0000844381	0.35464	1489.49
4200	1.69966(-8)	0.0000764846	0.344181	1548.81
4500	1.15857(-8)	0.0000544527	0.255928	1202.86
4700	8.85926(-9)	0.0000442963	0.221481	1107.41
5000	5.07767(-9)	0.0000264039	0.1373	713.962
5200	3.612(-9)	0.000019866	0.109263	600.946
5500	2.06644(-9)	0.0000117787	0.0671385	382.689
5700	1.40731(-9)	8.11312(-6)	0.0467721	269.641

Table 6: $|\psi(x) - x| < x\varepsilon$, $(x \geq e^b)$, $\varepsilon = C' X^{1/2} e^{-X}$

b	m	δ	ε	b	m	δ	ε
18.42	1	4.78(-4)	1.14790(-3)	900	22	2.08(-12)	2.39881(-11)
18.43	1	4.76(-4)	1.14336(-3)	950	21	2.15(-12)	2.36469(-11)
18.44	1	4.74(-4)	1.13884(-3)	1000	21	2.12(-12)	2.33114(-11)
18.45	1	4.71(-4)	1.13434(-3)	1050	21	2.09(-12)	2.29819(-11)
18.5	1	4.61(-4)	1.11208(-3)	1100	20	2.16(-12)	2.26511(-11)
18.7	1	4.22(-4)	1.02723(-3)	1150	20	2.13(-12)	2.23185(-11)
19.0	1	3.70(-4)	9.11615(-4)	1200	20	2.09(-12)	2.19902(-11)
19.5	1	2.96(-4)	7.46453(-4)	1250	19	2.17(-12)	2.16664(-11)
20	1	2.37(-4)	6.10561(-4)	1300	19	2.13(-12)	2.13331(-11)
21	1	1.52(-4)	4.07253(-4)	1350	19	2.10(-12)	2.10050(-11)
22	1	9.68(-5)	2.70618(-4)	1400	19	2.07(-12)	2.06828(-11)
23	1	6.17(-5)	1.79207(-4)	1450	18	2.14(-12)	2.03552(-11)
24	1	3.93(-5)	1.18314(-4)	1500	18	2.11(-12)	2.00268(-11)
25	1	2.51(-5)	7.79224(-5)	1550	18	2.07(-12)	1.97045(-11)
26	1	1.61(-5)	5.12515(-5)	1600	17	2.15(-12)	1.93836(-11)
27	1	1.06(-5)	3.37385(-5)	1650	17	2.12(-12)	1.90541(-11)
28	1	7.22(-6)	2.23274(-5)	1700	17	2.08(-12)	1.87301(-11)
29	1	5.26(-6)	1.49727(-5)	1750	17	2.05(-12)	1.84126(-11)
30	2	1.26(-6)	9.41428(-6)	1800	16	2.13(-12)	1.80866(-11)
35	2	1.22(-7)	1.05471(-6)	1850	16	2.09(-12)	1.77616(-11)
40	3	7.81(-9)	1.16290(-7)	1900	16	2.05(-12)	1.74427(-11)
45	4	5.60(-10)	1.23408(-8)	1950	15	2.14(-12)	1.71251(-11)
50	7	3.45(-11)	1.30541(-9)	2000	15	2.10(-12)	1.67987(-11)
75	26	2.20(-12)	3.32667(-11)	2100	15	2.02(-12)	1.61646(-11)
100	26	2.18(-12)	3.25398(-11)	2200	14	2.07(-12)	1.55206(-11)
150	26	2.16(-12)	3.13387(-11)	2300	13	2.12(-12)	1.48944(-11)
200	26	2.13(-12)	3.03713(-11)	2400	13	2.04(-12)	1.42535(-11)
250	25	2.18(-12)	2.95752(-11)	2500	12	2.10(-12)	1.36270(-11)
300	25	2.15(-12)	2.88982(-11)	2600	12	2.00(-12)	1.29976(-11)
350	25	2.13(-12)	2.83142(-11)	2700	11	2.06(-12)	1.23732(-11)
400	25	2.10(-12)	2.78000(-11)	3000	10	1.92(-12)	1.05303(-11)
450	24	2.16(-12)	2.73267(-11)	3200	9	1.86(-12)	9.32308(-12)
500	24	2.13(-12)	2.68923(-11)	3500	7	1.89(-12)	7.53761(-12)
550	24	2.10(-12)	2.64897(-11)	3700	6	1.83(-12)	6.39612(-12)
600	23	2.16(-12)	2.61010(-11)	4000	5	1.60(-12)	4.78674(-12)
650	23	2.14(-12)	2.57273(-11)	4500	3	1.23(-12)	2.46504(-12)
700	23	2.11(-12)	2.53666(-11)	4700	2	1.20(-12)	1.77229(-12)
750	22	2.17(-12)	2.50149(-11)	5000	2	6.51(-13)	9.76476(-13)
800	22	2.14(-12)	2.46639(-11)	5100	2	5.34(-13)	8.00754(-13)
850	22	2.11(-12)	2.43220(-11)	5200	2	4.38(-13)	6.56727(-13)

Table 7: η for the case $\varepsilon = C' X^{1/2} e^{-X}$

b	η_1	η_2	η_3	η_4
18.42	0.0211558	0.389901	7.18587	132.436
18.43	0.0210836	0.388781	7.16912	132.199
18.44	0.0210116	0.387664	7.15241	131.962
18.45	0.0209853	0.388227	7.18220	132.871
18.5	0.0207960	0.388884	7.27214	135.989
18.7	0.0195173	0.370829	7.04574	133.869
19.0	0.0177765	0.346641	6.75951	131.81
19.5	0.0149291	0.298581	5.97162	119.432
20	0.0128218	0.269258	5.65441	118.743
21	0.00895956	0.19711	4.33643	95.4014
22	0.00622421	0.143157	3.29261	75.73
23	0.00430097	0.103223	2.47736	59.4567
24	0.00295785	0.0739463	1.84866	46.2165
25	0.00202598	0.0526755	1.36956	35.6087
26	0.00138379	0.0373624	1.00878	27.2372
27	0.000944678	0.026451	0.740628	20.7376
28	0.000647495	0.0187774	0.544543	15.7918
29	0.000449182	0.0134755	0.404264	12.1279
30	0.0003295	0.0115325	0.403637	14.1273
35	0.0000421884	0.00168754	0.0675015	2.70006
40	5.23306(-6)	0.000235488	0.0105969	0.476863
45	6.17042(-7)	0.0000308521	0.00154261	0.0771303
50	9.79061(-8)	7.34296(-6)	0.000550722	0.0413041
75	3.32667(-9)	3.32667(-7)	0.0000332667	0.00332667
100	4.88096(-9)	7.32145(-7)	0.000109822	0.0164733
150	6.26774(-9)	1.25355(-6)	0.00025071	0.0501419
200	7.59281(-9)	1.8982(-6)	0.000474551	0.118638
250	8.87255(-9)	2.66177(-6)	0.00079853	0.239559
300	1.01144(-8)	3.54003(-6)	0.00123901	0.433654
350	1.13257(-8)	4.53027(-6)	0.00181211	0.724843
400	1.251(-8)	5.62949(-6)	0.00253327	1.13997
450	1.36634(-8)	6.83168(-6)	0.00341584	1.70792
500	1.47907(-8)	8.13491(-6)	0.0044742	2.46081
550	1.58938(-8)	9.5363(-6)	0.00572178	3.43307
600	1.69656(-8)	0.0000110277	0.00716799	4.65919
650	1.80091(-8)	0.0000126064	0.00882445	6.17712
700	1.90249(-8)	0.0000142687	0.0107015	8.02615
750	2.0012(-8)	0.0000160096	0.0128076	10.2461
800	2.09643(-8)	0.0000178197	0.0151467	12.8747
850	2.18898(-8)	0.0000197008	0.0177307	15.9577

Table 8: η for the case $\varepsilon = C' X^{1/2} e^{-X}$

b	η_1	η_2	η_3	η_4
900	2.27887(-8)	0.0000216493	0.0205668	19.5385
950	2.36469(-8)	0.0000236469	0.0236469	23.6469
1000	2.4477(-8)	0.0000257009	0.0269859	28.3352
1050	2.52801(-8)	0.0000278081	0.0305889	33.6478
1100	2.60488(-8)	0.0000299561	0.0344495	39.617
1150	2.67822(-8)	0.0000321386	0.0385663	46.2796
1200	2.74877(-8)	0.0000343597	0.0429496	53.687
1250	2.81663(-8)	0.0000366162	0.0476011	61.8814
1300	2.87997(-8)	0.0000388796	0.0524875	70.8582
1350	2.94071(-8)	0.0000411699	0.0576378	80.693
1400	2.999(-8)	0.0000434855	0.063054	91.4283
1450	3.05328(-8)	0.0000457993	0.0686989	103.048
1500	3.10416(-8)	0.0000481145	0.0745774	115.595
1550	3.15272(-8)	0.0000504435	0.0807096	129.135
1600	3.1983(-8)	0.000052772	0.0870738	143.672
1650	3.2392(-8)	0.0000550663	0.0936128	159.142
1700	3.27777(-8)	0.000057361	0.100382	175.668
1750	3.31427(-8)	0.0000596569	0.107382	193.288
1800	3.34602(-8)	0.0000619014	0.114518	211.857
1850	3.3747(-8)	0.0000641193	0.121827	231.471
1900	3.40133(-8)	0.0000663259	0.129336	252.204
1950	3.42503(-8)	0.0000685005	0.137001	274.002
2000	3.52773(-8)	0.0000740823	0.155573	326.703
2100	3.55621(-8)	0.0000782366	0.172121	378.665
2200	3.56974(-8)	0.0000821041	0.188839	434.331
2300	3.57465(-8)	0.0000857917	0.2059	494.16
2400	3.56337(-8)	0.0000890842	0.22271	556.776
2500	3.54303(-8)	0.0000921187	0.239509	622.722
2600	3.50936(-8)	0.0000947527	0.255832	690.747
2700	3.71195(-8)	0.000111358	0.334075	1002.23
3000	3.3697(-8)	0.00010783	0.345057	1104.18
3200	3.26308(-8)	0.000114208	0.399727	1399.04
3500	2.8266(-8)	0.000105998	0.397491	1490.59
3750	2.4491(-8)	0.0000979639	0.391855	1567.42
4000	2.01043(-8)	0.0000844381	0.35464	1489.49
4200	1.69978(-8)	0.0000764902	0.344206	1548.93
4500	1.15857(-8)	0.0000544527	0.255928	1202.86
4700	8.86144(-9)	0.0000443072	0.221536	1107.68
5000	4.98003(-9)	0.0000253981	0.12953	660.606
5100	4.16392(-9)	0.0000216524	0.112592	585.48
5200	3.42352(-9)	0.0000178468	0.0930354	484.993

Table 9: η for the case $\varepsilon = C' X^{1/2} e^{-X}$

b	m	δ	ε	η_1	η_2	η_3	η_4
3800	6	1.67(-12)	5.86122(-12)	2.23312(-8)	0.000085082	0.324163	1235.06
3810	5	1.94(-12)	5.80739(-12)	2.21842(-8)	0.0000847437	0.323721	1236.61
3820	5	1.92(-12)	5.74859(-12)	2.20171(-8)	0.0000843255	0.322967	1236.96
3830	5	1.90(-12)	5.69039(-12)	2.18511(-8)	0.0000839082	0.322207	1237.28
3840	5	1.88(-12)	5.63277(-12)	2.16862(-8)	0.0000834918	0.321443	1237.56
3850	5	1.86(-12)	5.57575(-12)	2.15224(-8)	0.0000830764	0.320675	1237.8
3860	5	1.84(-12)	5.51930(-12)	2.13597(-8)	0.000082662	0.319902	1238.02
3870	5	1.82(-12)	5.46344(-12)	2.11982(-8)	0.0000822488	0.319126	1238.21
3880	5	1.80(-12)	5.40817(-12)	2.10378(-8)	0.0000818369	0.318346	1238.36
3890	5	1.78(-12)	5.35348(-12)	2.08786(-8)	0.0000814265	0.317563	1238.5
3900	5	1.77(-12)	5.29927(-12)	2.07201(-8)	0.0000810158	0.316772	1238.58
3910	5	1.75(-12)	5.24559(-12)	2.05627(-8)	0.0000806058	0.315975	1238.62
3920	5	1.73(-12)	5.19249(-12)	2.04065(-8)	0.0000801975	0.315176	1238.64
3930	5	1.71(-12)	5.13998(-12)	2.02515(-8)	0.000079791	0.314376	1238.64
3940	5	1.70(-12)	5.08798(-12)	2.00975(-8)	0.0000793852	0.313572	1238.61
3950	5	1.68(-12)	5.03642(-12)	1.99442(-8)	0.0000789791	0.312757	1238.52
3960	5	1.66(-12)	4.98545(-12)	1.97922(-8)	0.0000785752	0.311944	1238.42
3970	5	1.64(-12)	4.93509(-12)	1.96417(-8)	0.0000781739	0.311132	1238.31
3980	5	1.63(-12)	4.88504(-12)	1.94913(-8)	0.0000777704	0.310304	1238.11
3990	5	1.61(-12)	4.83561(-12)	1.93424(-8)	0.0000773697	0.309479	1237.92
4000	5	1.60(-12)	4.78674(-12)	1.91948(-8)	0.0000769713	0.308655	1237.71